IV. *On some Definite Integrals, and a New Method of Reducing a Function of Spherical Co-ordinates to a Series of Spherical Harmonics.*

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§ 1. *Introductory.*

THE following investigation deals with some definite integrals which are useful when it is desired to express a function of two angular variables by means of a series of spherical surface harmonics. An important theorem concerning these integrals leads to a method which considerably reduces the arithmetical labour involved in the reductions, and secures in practice the advantage of obtaining the numerical values of the coefficients of lower degrees independently of those of higher degrees.

The zonal harmonic of degree n is denoted by P_n and defined as usual by

$$P_n = \frac{1}{2^n 1 \cdot 2 \ldots n} \frac{d^n}{d\mu^n} (\mu^2 - 1)^n .$$

The tesseral harmonic of degree n and type σ is denoted by

$$T_n^\sigma = Q_n^\sigma (g_n^\sigma \cos \sigma\phi + h_n^\sigma \sin \sigma\phi),$$

where

$$Q_n^\sigma = \sin^\sigma \theta \frac{d^\sigma P_n}{d\mu^\sigma} .$$

In these equations θ represents an angle measured on a sphere from a point as pole and $\mu = \cos \theta$. The longitude measured from some standard meridian is denoted by ϕ.

The name given by HEINE, and translated by TODHUNTER as "Associated Functions of the First Kind," is too cumbersome for use, and I propose to call these functions "Tesseral Functions." The tesseral function is converted into a tesseral harmonic by the factor cos $\sigma\phi$ or sin $\sigma\phi$. The name may not perhaps appear to be appropriate, because it is only the factor which gives the function its "tesseral" character, but it is short and suggests at once the function it denotes.

The present investigation deals in great part with the definite integrals, taken over the surface of a sphere of unit radius, of the product of two tesseral functions and of the product of a tesseral function and cos $p\theta$ or sin $p\theta$, the tesseral functions being all referred to the same axis. I shall denote, as usual, the factorial product of the numbers up to n by $n\,!$, but have found it necessary to introduce a separate notation for the products of successive even or successive odd numbers. I consequently define

$$n\,!\,! = n\,.\,(n-2)\,!\,!\,.$$

Starting with $$1\,!\,! = 1, \qquad 2\,!\,! = 2,$$

it follows that, for positive values of n,

$$n\,!\,! = n\,.\,n-2\,.\,n-4\,\ldots\,,$$

where the last factor is either 1 or 2, according as n is odd or even.

We may extend the definition to negative values of the argument, for the successive substitution of $n = 1$, $n = -1$, $n = -3$, into the first equation, gives

$$(-1)\,!\,! = 1, \qquad (-3)\,!\,! = -1, \qquad (-5)\,!\,! = \tfrac{1}{3},$$

and generally if n is negative and odd,

$$n\,!\,! = (-1)^{\frac{-n-1}{2}} \frac{1}{(-n-2)\,!\,!}\,.$$

For $n = 2$, the original equation gives $0\,!\,! = 1$,

for $n = 0$, $$(-2)\,!\,! = \infty\,,$$

and similarly for all negative and even values of n, $n\,!\,!$ is infinite. The ratio of two of these factorials of negative numbers is, however, finite, for it is easily shewn that if m and n be two negative numbers, whether even or odd,

$$\frac{m\,!\,!}{n\,!\,!} = (-1)^{\frac{n-m}{2}} \frac{(-n-2)\,!\,!}{(-m-2)\,!\,!}\,.$$

One of the advantages of a separate notation for what may be called the "alternate" or "double" factorial, is due to the fact that it often saves the inconvenience of different expressions for odd and even numbers.

§ 2. *Formulæ of Transformation.*

It is convenient to collect together some equations which will often be required. Most of these equations will be found already in previous writings, such as HEINE'S Treatise or ADAMS' "Researches in Terrestrial Magnetism."

As regards zonal harmonics, it is only necessary to quote the well-known relations ;

$$(2n+1)\,\mu \mathrm{P}_n = (n+1)\,\mathrm{P}_{n+1} + n\mathrm{P}_{n-1} \quad \ldots\ldots \quad (1),$$

$$(2n+1)\,\mathrm{P}_n = \frac{d\mathrm{P}_{n+1}}{d\mu} - \frac{d\mathrm{P}_{n-1}}{d\mu} \quad \ldots\ldots \quad (2).$$

From these equations we derive the following:

$$(2n+1)\,\mu \mathrm{Q}_n^\sigma = (n-\sigma+1)\,\mathrm{Q}_{n+1}^\sigma + (n+\sigma)\,\mathrm{Q}_{n-1}^\sigma \quad \ldots\ldots \quad (\mathrm{A}),$$

$$(2n+1)\sin\theta\,\mathrm{Q}_n^\sigma = \mathrm{Q}_{n+1}^{\sigma+1} - \mathrm{Q}_{n-1}^{\sigma+1} \quad \ldots\ldots \quad (\mathrm{B}),$$

$$= (n+\sigma)(n+\sigma-1)\,\mathrm{Q}_{n-1}^{\sigma-1} - (n-\sigma+2)(n-\sigma+1)\,\mathrm{Q}_{n+1}^{\sigma-1}. \quad (\mathrm{C}),$$

$$\frac{2\sigma \mathrm{Q}_n{}^\sigma}{\sin\theta} = (n+\sigma)(n+\sigma-1)\,\mathrm{Q}_{n-1}^{\sigma-1} + \mathrm{Q}_{n-1}^{\sigma+1} \quad \ldots\ldots \quad (\mathrm{D}),$$

$$= \mathrm{Q}_{n+1}^{\sigma+1} + (n-\sigma+2)(n-\sigma+1)\,\mathrm{Q}_{n+1}^{\sigma-1} \quad \ldots\ldots \quad (\mathrm{E}),$$

$$\mathrm{Q}_n^\sigma - \mathrm{Q}_{n-2}^\sigma = (n+\sigma-2)(n+\sigma-3)\,\mathrm{Q}_{n-2}^{\sigma-2} - (n-\sigma+2)(n-\sigma+1)\,\mathrm{Q}_n^{\sigma-2}. \quad (\mathrm{F}),$$

$$(2n+1)\frac{d}{d\mu}\sin^\rho\theta\,\frac{d^\sigma \mathrm{P}_n}{d\mu^\sigma}$$

$$= (n+\sigma)(n+\sigma-\rho+1)\sin^{\rho-2}\theta\,\frac{d^\sigma \mathrm{P}_{n-1}}{d\mu^\sigma} - (n-\sigma+1)(n-\sigma+\rho)\sin^{\rho-2}\theta\,\frac{d^\sigma \mathrm{P}_{n+1}}{d\mu^\sigma}. \quad (\mathrm{G}),$$

$$2\sigma\frac{d}{d\mu}\sin^\rho\theta\,\frac{d^\sigma \mathrm{P}_n}{d\mu^\sigma} = (2\sigma-\rho)\sin^\rho\theta\,\frac{d^{\sigma+1}\mathrm{P}_n}{d\mu^{\sigma+1}} - \rho(n+\sigma)(n-\sigma+1)\sin^{\rho-2}\theta\,\frac{d^{\sigma-1}\mathrm{P}_n}{d\mu^{\sigma-1}}. \quad (\mathrm{H}).$$

As special cases of (G) and (H) we may put ρ equal successively to σ, $\sigma+1$, and $\sigma+2$. The following equations are thus obtained:

$$(2n+1)\sin^2\theta\,\frac{d}{d\mu}\mathrm{Q}_n^\sigma = (n+1)(n+\sigma)\,\mathrm{Q}_{n-1}^\sigma - n(n+1-\sigma)\,\mathrm{Q}_{n+1}^\sigma. \quad (\mathrm{G}_1),$$

$$2\sin\theta\,\frac{d}{d\mu}\mathrm{Q}_n^\sigma = \mathrm{Q}_n^{\sigma+1} - (n+\sigma)(n-\sigma+1)\,\mathrm{Q}_n^{\sigma-1} \quad \ldots\ldots \quad (\mathrm{H}_1),$$

$$(2n+1)\sin\theta\,\frac{d}{d\mu}\sin\theta\,\mathrm{Q}_n^\sigma = n(n+\sigma)\,\mathrm{Q}_{n-1}^\sigma - (n+1)(n+1-\sigma)\,\mathrm{Q}_{n+1}^\sigma. \quad (\mathrm{G}_2),$$

$$2\sigma\frac{d}{d\mu}\sin\theta\,\mathrm{Q}_n^\sigma = (\sigma-1)\,\mathrm{Q}_n^{\sigma+1} - (\sigma+1)(n+\sigma)(n-\sigma+1)\,\mathrm{Q}_n^{\sigma-1}. \quad (\mathrm{H}_2),$$

$$(2n+1)\frac{d}{d\mu}\sin^2\theta\,\mathrm{Q}_n^\sigma = (n-1)(n+\sigma)\,\mathrm{Q}_{n-1}^\sigma - (n+2)(n-\sigma+1)\,\mathrm{Q}_{n+1}^\sigma \quad (\mathrm{G}_3),$$

$$2\sigma\frac{d}{d\mu}\sin^2\theta\,\mathrm{Q}_n^\sigma = (\sigma-2)\sin\theta\,\mathrm{Q}_n^{\sigma+1} - (\sigma+2)(n+\sigma)(n-\sigma+1)\sin\theta\,\mathrm{Q}_n^{\sigma-1} \quad (\mathrm{H}_3).$$

The formula (A) is well known and may be obtained by σ differentiations of (1), substituting in the result an equation derived from $\sigma-1$ differentiations of (2).

Equation (B) is the result of σ differentiations of (2), and multiplication by $\sin^{\sigma+1}\theta$. (C) may be proved by combining

$$\frac{d^\sigma}{d\mu^\sigma}(1-\mu^2)\frac{dP_n}{d\mu} = (1-\mu^2)\frac{d^{\sigma+1}P_n}{d\mu^{\sigma+1}} - 2\mu\sigma\frac{d^\sigma P_n}{d\mu^\sigma} - \sigma \,.\, \sigma - 1\,\frac{d^{\sigma-1}P_n}{d\mu^{\sigma-1}}$$

with the fundamental equation for zonal harmonics. The latter leads directly to

$$\frac{d^\sigma}{d\mu^\sigma}(1-\mu^2)\frac{dP_n}{d\mu} = -\frac{d^{\sigma-1}}{d\mu^{\sigma-1}}\,n(n+1)\,.\,P_n$$

from which we derive (by equating the two expressions on the right-hand sides):

$$(1-\mu^2)\frac{d^{\sigma+1}P_n}{d\mu^{\sigma+1}} = 2\mu\sigma\frac{d^\sigma P_n}{d\mu^\sigma} - (n+\sigma)(n-\sigma+1)\frac{d^{\sigma-1}P_n}{d\mu^{\sigma-1}},$$

or, after multiplication by $(1-\mu^2)^{\frac{\sigma}{2}}$,

$$\sin\theta\, Q_n^{\sigma+1} = 2\mu\sigma\, Q_n^\sigma - (n+\sigma)(n-\sigma+1)\sin\theta\, Q_n^{\sigma-1}.$$

If μQ_n^σ be now substituted from (A) and $\sin\theta\, Q_n^{\sigma-1}$ from (B), the equation (C) is obtained.

If $\sigma - 1$ be written for σ in (B) and $\sigma + 1$ for σ in (C) we may combine the two equations, so as to give (D) and (E). (F) is an important relation obtained from (B) and (C). The formulæ (G) and (H) are easily derived by direct differentiation and a few simple transformations.

§ 3. $\int_{-1}^{+1} Q_n^\sigma\, d\mu$.

If the equation (H_2) is integrated with respect to μ between the limits -1 and $+1$, the left-hand side vanishes at both limits: hence, after changing from $\sigma + 1$ to σ,

$$\int_{-1}^{+1} Q_n^\sigma\, d\mu = \frac{\sigma}{\sigma-2}(n+\sigma-1)(n-\sigma+2)\int_{-1}^{+1} Q_n^{\sigma-2}\, d\mu,$$

and, by applying the same process to the right-hand side,

$$= \frac{\sigma}{\sigma-4}(n+\sigma-1)(n+\sigma-3)(n-\sigma+2)(n-\sigma+4)\int_{-1}^{+1} Q_n^{\sigma-4}\, d\mu.$$

Repetition of the same proceeding will ultimately lead for even values of σ to

$$\int_{-1}^{+1} Q_n^\sigma\, d\mu = \frac{\sigma}{2}\,\frac{(n+\sigma-1)!!\,(n-2)!!}{(n-\sigma)!!\,\;(n+1)!!}\int_{-1}^{+1} Q_n^2\, d\mu.$$

But

$$\int_{-1}^{+1} Q_n^2\, d\mu = \int_{-1}^{+1}(1-\mu^2)\frac{d^2 P_n}{d\mu^2}\, d\mu = 2\int_{-1}^{+1}\mu\frac{dP_n}{d\mu}\, d\mu = 2\int_{-1}^{+1}\left(\frac{d\mu P_n}{d\mu} - P_n\right) d\mu = 2\Big[\mu P_n\Big]_{-1}^{+1},$$

where the special case $n = 0$, for which the integral vanishes, is excluded.

Hence

$$\int_{-1}^{+1} Q_n^2\, d\mu = 4, \text{ if } n \text{ be even}; \; = 0, \text{ if } n \text{ be odd};$$

and finally

$$\int_{-1}^{+1} Q_n^\sigma\, d\mu = 2\sigma\frac{(n+\sigma-1)!!}{(n-\sigma)!!}\frac{(n-2)!!}{(n+1)!!}, \text{ if } \sigma \text{ and } n \text{ be even};$$

$$= 0, \text{ if } \sigma \text{ be even and } n \text{ odd}.$$

For odd values of σ, it is more convenient to use formula (G_3), which leads to:

$$\int_{-1}^{+1} Q_n^\sigma\, d\mu = \frac{(n+\sigma-1)(n-2)}{(n-\sigma)(n+1)}\int Q_{n-2}^\sigma\, d\mu.$$

By repetition of the same process, we get for even values of n ultimately

$$\int_{-1}^{+1} Q_n^\sigma\, d\mu = \frac{(n+\sigma-1)!!(n-2)!!}{(n-\sigma)!!(n+1)!!}\frac{\sigma!!}{(\sigma-3)!!}\frac{1}{(2\sigma-2)!!}\int_{-1}^{+1} Q_{\sigma-1}^\sigma\, d\mu.$$

As the expression under the integral sign of the right-hand side vanishes, the value of the integral on the left-hand side is zero.

When n is odd we are ultimately led to

$$\int_{-1}^{+1} Q_n^\sigma\, d\mu = \frac{(n+\sigma-1)!!(n-2)!!}{(n-\sigma)!!(n+1)!!}\frac{(\sigma+1)!!}{(\sigma-2)!!}\frac{1}{(2\sigma-1)!!}\int_{-1}^{+1} Q_\sigma^\sigma\, d\mu.$$

But

$$\int_{-1}^{+1} Q_\sigma^\sigma\, d\mu = (2\sigma-1)!!\int_{-1}^{+1}\sin^\sigma\theta\, d\mu = (2\sigma-1)!!\int_0^\pi \sin^{\sigma+1}\theta\, d\theta = \frac{\sigma!!}{(\sigma+1)!!}(2\sigma-1)!!\,\pi.$$

Collecting the results, we may put generally

$$\int_{-1}^{+1} Q_n^\sigma\, d\mu = c\sigma\frac{(n+\sigma-1)!!(n-2)!!}{(n-\sigma)!!(n+1)!!}, \text{ if } n+\sigma \text{ be even},$$

where c is equal to 2 or π according as σ be even or odd. When $(n+\sigma)$ is odd, the integral vanishes.

The results of this paragraph allow us to represent a quantity which is constant over a sphere, in terms of a series of tesseral functions which are all of the same type, the type being arbitrary. It follows that we may express $\cos\sigma\lambda$ and $\sin\sigma\lambda$ in terms of a series of tesseral harmonics of type σ.

To find the values of the coefficients of such a series we put

$$\cos \sigma\phi = \mathrm{A}_\sigma^\sigma \mathrm{T}_\sigma^\sigma + \mathrm{A}_{\sigma+2}^\sigma \mathrm{T}_{\sigma+2}^\sigma + \ldots \mathrm{A}_n^\sigma \mathrm{T}_n^\sigma + \ldots$$

and integrating over unit sphere, after multiplying as usual by T_n^σ, we find

$$\int_0^{2\pi}\int_{-1}^{+1} \mathrm{Q}_n^\sigma \cos^2 \sigma\phi \, d\phi \, d\mu = \mathrm{A}_n^\sigma \int_0^{2\pi}\int_{-1}^{+1} (\mathrm{T}_n^\sigma)^2 \, d\phi \, d\mu = \frac{2\pi}{2n+1} \cdot \frac{(n+\sigma)\,!}{(n-\sigma)\,!} \mathrm{A}_n^\sigma.$$

Substituting on the left-hand side the integral found above we obtain

$$\mathrm{A}_n^\sigma = c\sigma \frac{2n+1}{2} \cdot \frac{(n-\sigma)\,!}{(n+\sigma)\,!} \cdot \sigma \frac{(n+\sigma-1)\,!\,!\,(n-2)\,!\,!}{(n-\sigma)\,!\,!\,(n+1)\,!\,!} = c\sigma \frac{2n+1}{2} \cdot \frac{(n-\sigma-1)\,!\,!\,(n-2)\,!\,!}{(n+\sigma)\,!\,!\,(n+1)\,!\,!},$$

where c is 2 or π according as σ and n are both even or both odd.

We obtain, for instance, in this way for $\sigma = 2$,

$$\tfrac{1}{2} = \frac{5}{1\,.\,2\,.\,3\,.\,4} \mathrm{Q}_2^{\,2} + \frac{9}{3\,.\,4\,.\,5\,.\,6} \mathrm{Q}_4^{\,2} + \frac{13}{5\,.\,6\,.\,7\,.\,8} \mathrm{Q}^2_6 + \ldots$$

§ 4. $\int_{-1}^{+1} \mathrm{Q}_n^\sigma \sin^\lambda \theta \, d\mu, \qquad \int_{-1}^{+1} \mu \mathrm{Q}_n^\sigma \sin^\lambda\theta \, d\mu.$

If we put $\rho = \sigma + \lambda + 2$ in equation (G), and after changing from $n + 1$ to n, integrate both sides, we obtain:

$$\int_{-1}^{+1} \sin^\lambda \theta \, \mathrm{Q}_n^\sigma \, d\mu = \frac{(n+\sigma-1)\,(n-\lambda-2)}{(n-\sigma)\,(n+\lambda+1)} \int_{-1}^{+1} \sin^\lambda \theta \, \mathrm{Q}_{n-2}^\sigma \, d\mu$$

$$= \frac{(n+\sigma-1)\,(n+\sigma-3)\,(n-\lambda-2)\,(n-\lambda-4)}{(n-\sigma)\,(n-\sigma-2)\,(n+\lambda+1)\,(n+\lambda-1)} \int_{-1}^{+1} \sin^\lambda \theta \, \mathrm{Q}_{n-4}^\sigma \, d\mu.$$

If $n - \sigma$ be odd, a continuation of this process reduces the degree of the tesseral function on the right-hand side until it becomes smaller than σ, and this will happen before any of the factors in the denominator become zero, so that in that case the integral on the left-hand side is zero.

If $n - \sigma$ be even, the integral on the right-hand side ultimately becomes

$$\int_{-1}^{+1} \sin^\lambda \theta \, \mathrm{Q}_\sigma^\sigma \, d\mu = (2\sigma - 1)\,!\,! \int_0^\pi \sin^{(\lambda+\sigma+1)} \theta \, d\theta = (2\sigma - 1)\,!\,! \frac{(\lambda+\sigma)\,!\,!}{(\lambda+\sigma+1)\,!\,!} c,$$

where c is 2 or π according as $\lambda + \sigma$ is even or odd.

The integral to be determined now becomes

$$\int_{-1}^{+1} \mathrm{Q}_n^\sigma \sin^\lambda \theta \, d\mu = c \frac{(n+\sigma-1)\,!\,!\,(\sigma+\lambda)\,!\,!}{(n-\sigma)\,!\,!\,(n+\lambda+1)\,!\,!} [(n-\lambda-2)\,(n-\lambda-4) \ldots (\sigma-\lambda)]$$ (if $n - \sigma$ be even)

$$= 0, \text{ if } n - \sigma \text{ be odd.}$$

A little care is necessary in the interpretation of the square bracket on the right-hand side. All the factors of the product contained in it may be positive or negative, but when λ is intermediate between n and σ, some may be positive and some negative. In the latter case, one of the factors will be zero when $\sigma - \lambda$ is even, and in that case the integral on the right-hand side is zero. The above expression does not include the special case $n = \sigma$. By extending the notation of double factorials to negative numbers as defined in § 1, we may also write, including the cases when $n - \lambda$, or both $n - \lambda$ and $\sigma - \lambda$, are negative, or when $n = \sigma$,

$$\int_{-1}^{+1} \mathrm{Q}_n^\sigma \sin^\lambda \theta \, d\mu = c \frac{(n + \sigma - 1)!!(\sigma + \lambda)!!(n - \lambda - 2)!!}{(n - \sigma)!!(\sigma - \lambda - 2)!!(n + \lambda + 1)!!} \text{ if } n - \sigma \text{ be even,}$$

$$= 0 \text{ if } n - \sigma \text{ be odd.}$$

The integral $\int_{-1}^{+1} \mu \mathrm{Q}_n^\sigma \sin^\lambda \theta \, d\mu$ reduces to the one just determined with the help of equation (A). We thus find:

$$\int_{-1}^{+1} \mu \mathrm{Q}_n^\sigma \sin^\lambda \theta \, d\mu = c \frac{(n + \sigma)!!(\sigma + \lambda)!!(n - \lambda - 3)!!}{(n - \sigma - 1)!!(\sigma - \lambda - 2)!!(n + \lambda + 2)!!} \quad \text{if } n - \sigma \text{ be odd,}$$

$$= 0 \text{ if } n - \sigma \text{ be even.}$$

The factor c takes, as before, the value 2 or π according as $\sigma + \lambda$ is even or odd.

For the special case $\sigma = 0$, the tesseral harmonics reduce to zonal harmonics, and the last equation becomes

$$\int_{-1}^{+1} \mu \mathrm{P}_n \sin^\lambda \theta \, d\mu = c \frac{n!!\,\lambda!!\,(n - \lambda - 3)!!}{(n - 1)!!(-\lambda - 2)!!(n + \lambda + 2)!!} \quad \text{if } n \text{ be odd.}$$

If λ be even and $n > \lambda + 2$, the fraction on the right-hand side is zero, because in that case $(-\lambda - 2)!! = \infty$, and the numerator remains finite, but if $n < \lambda + 2$, the value of $\frac{(n - \lambda - 3)!!}{(-\lambda - 2)!!}$ remains finite whether λ be even or odd, and, in that case, the right-hand side may also be written (avoiding negative arguments of the factorials):

$$= (-)^{\frac{n-1}{2}} c \frac{n!!\lambda!!\lambda!!}{(n - 1)!!(n + \lambda + 2)!!(\lambda - n + 1)!!}.$$

If n be odd and $n \geqq \lambda + 2$, we may transform the negative factorial and write the value of the integral

$$= (-1)^{\frac{\lambda+1}{2}} \pi \frac{n!!\lambda!!\lambda!!(n - \lambda - 3)!!}{(n - 1)!!(n + \lambda + 2)!}.$$

Similarly we obtain

$$\int_{-1}^{+1} \mathrm{P}_n \sin^\lambda \theta \, d\mu = c \, \frac{(n-1)!!\,\lambda!!\,(n-\lambda-2)!!}{n!!\,(-\lambda-2)!!\,(n+\lambda+1)!!}, \quad \text{if } n \text{ be even, other-}$$

wise the integral is zero.

This includes, as particular cases,

$$\int_{-1}^{+1} \mathrm{P}_n \sin^\lambda \theta \, d\mu = 0 \ \text{ if } \lambda \text{ be even and } n > \lambda + 1,$$

$$= (-1)^{\frac{\lambda+1}{2}} \pi \frac{(n-1)!!\,\lambda!!\,\lambda!!\,(n-\lambda-2)!!}{n!!\,(n+\lambda+1)!!} \quad \text{if } \lambda \text{ be odd and } n \geqq \lambda + 1.$$

$$= (-1)^{\frac{n}{2}} c \frac{(n-1)!!\,\lambda!!\,\lambda!!}{n!!\,(\lambda+n+1)!!\,(\lambda-n)!!} \quad \text{if } n < \lambda + 1.$$

Dr. W. D. Niven ('Phil. Trans.,' vol. 170 (1879 I.), p. 379) has already obtained an expression for the integral $\int_{-1}^{+1} \mathrm{P}_n \sin^\lambda \theta \, d\mu$. His results are identical with the above, when allowing for the difference in the notation and after correcting an obvious misprint in the equation marked (16) on p. 388.

§ 5. $\int_{-1}^{+1} \mathrm{P}_\epsilon \frac{d^\rho \mathrm{P}_n}{d\mu^\rho} \, d\mu$.

If we write down the differential coefficients of P_n by means of the equation

$$d\mathrm{P}_n/d\mu = (2n-1)\,\mathrm{P}_{n-1} + (2n-5)\,\mathrm{P}_{n-3} + \ldots,$$

and repeat the same process ρ times, we obtain a series beginning with

$$d^\rho \mathrm{P}_n/d\mu^\rho = (2n-1)\,(2n-3)\ldots(2n-2\rho+1)\,\mathrm{P}_{n-\rho} + \ldots$$

There being no term containing a zonal harmonic of higher degree than $n - \rho$, we conclude that the above integral vanishes when $\epsilon > n - \rho$, and that when $\epsilon = n - \rho$

$$\int_{-1}^{+1} \mathrm{P}_\epsilon \frac{d^\rho \mathrm{P}_n}{d\mu^\rho} \, d\mu = \frac{2\,(2n-1)!!}{(2n-2\rho+1)!!}.$$

If $\epsilon < n - \rho$ we may transform the integral as follows :

$$\int_{-1}^{+1} \mathrm{P}_\epsilon \frac{d^\rho \mathrm{P}_n}{d\mu^\rho} \, d\mu = \frac{1}{2^\epsilon \epsilon!} \int_{-1}^{+1} \frac{d^\epsilon (\mu^2-1)^\epsilon}{d\mu^\epsilon} \cdot \frac{d^\rho \mathrm{P}_n}{d\mu^\rho} \, d\mu.$$

After ϵ partial integrations, in which the first term always vanishes at both limits, the integral becomes

$$\frac{(-1)^\epsilon}{2^\epsilon \epsilon!} \int_{-1}^{+1} (\mu^2-1)^\epsilon \frac{d^{\rho+\epsilon} \mathrm{P}_n}{d\mu^{\rho+\epsilon}} = \frac{1}{(2\epsilon)!!} \int_{-1}^{+1} \sin^{\epsilon-\rho} \theta \mathrm{Q}_n{}^{\rho+\epsilon} \, d\mu.$$

The integral on the right-hand side has been found in the last paragraph. Writing $\rho + \epsilon = \sigma$ and $\epsilon - \rho = \lambda$, we note that $(n - \lambda - 2) = (n - \epsilon + \rho - 2)$ is necessarily positive as $\epsilon < n - \rho$, and that $\sigma - \lambda$ is positive and even. If, further, $n - \sigma$ be even $n - \lambda$ must be even also, because the difference between these quantities is $\sigma - \lambda = 2\rho$. We may now write the value of the integral

$$\int_{-1}^{+1} \mathrm{P}_\epsilon \frac{d^\rho \mathrm{P}_n}{d\mu^\rho}\, d\mu = 2\, \frac{(n+\epsilon+\rho-1)!!\,(n-\epsilon+\rho-2)!!}{(n-\epsilon-\rho)!!\,(n+\epsilon-\rho+1)!!}\, \frac{1}{(2\rho-2)!!}$$

$$\text{if } \epsilon \lesseqqgtr n - \rho \text{ and } n + \rho + \epsilon \text{ is even,}$$

$$= 0, \text{ in all other cases.}$$

If $\rho = 1$, the above reduces to

$$\int_{-1}^{+1} \mathrm{P}_\epsilon \frac{d\mathrm{P}_n}{d\mu}\, d\mu = 2.$$

§ 6. $\int_{-1}^{+1} \mathrm{Q}_n^{\sigma+2}\, \mathrm{Q}_i^\sigma\, d\mu$, $\int_{-1}^{+1} \mathrm{Q}_n^\sigma\, \mathrm{Q}_n^\rho\, d\mu$, $\int_{-1}^{+1} \mathrm{Q}_n^\sigma\, \mathrm{Q}_{n-2}^\rho\, d\mu$.

Before discussing the general integral of the product of two tesseral functions, it is convenient to obtain the solution in a few special cases. When the type of the two tesserals differs by 2, we may transform the integrals as follows:

$$\int_{-1}^{+1} (1-\mu^2)^{\frac{\sigma+2}{2}} \frac{d^{\sigma+2}\mathrm{P}_n}{d\mu^{\sigma+2}} (1-\mu^2)^{\frac{\sigma}{2}} \frac{d^\sigma \mathrm{P}_i}{d\mu^\sigma}\, d\mu$$

$$= \int_{-1}^{+1} (1-\mu^2)^{\sigma+1} \frac{d^{\sigma+2}\mathrm{P}_n}{d\mu^{\sigma+2}} \cdot \frac{d^\sigma \mathrm{P}_i}{d\mu^\sigma}\, d\mu$$

$$= \frac{1}{2i+1} \int_{-1}^{+1} (1-\mu^2)^{\sigma+1} \frac{d^{\sigma+2}\mathrm{P}_n}{d\mu^{\sigma+2}} \left(\frac{d^{\sigma+1}\mathrm{P}_{i+1}}{d\mu^{\sigma+1}} - \frac{d^{\sigma+1}\mathrm{P}_{i-1}}{d\mu^{\sigma+1}} \right) d\mu.$$

By the application of RODRIGUEZ's theorem the right-hand side reduces to

$$\frac{(-1)^{\sigma+1}}{2i+1} \frac{(i+\sigma)!}{(i-\sigma)!} \int_{-1}^{+1} \frac{d^{\sigma+2}\mathrm{P}_n}{d\mu^{\sigma+2}} \left[(i+\sigma+2)(i+\sigma+1) \frac{d^{-\sigma-1}\mathrm{P}_{i+1}}{d\mu^{\sigma+1}} - (i-\sigma)(i-\sigma-1) \frac{d^{-\sigma-1}\mathrm{P}_{i-1}}{d\mu^{\sigma+1}} \right] d\mu.$$

Integrating each term partially $\sigma + 1$ times we arrive at the equation

$$\int_{-1}^{+1} \mathrm{Q}_n^{\sigma+2}\, \mathrm{Q}_i^\sigma\, d\mu$$

$$= \frac{1}{2i+1} \frac{(i+\sigma)!}{(i-\sigma)!} \int_{-1}^{+1} \frac{d\mathrm{P}_n}{d\mu} \left[(i+\sigma+2)(i+\sigma+1)\, \mathrm{P}_{i+1} - (i-\sigma)(i-\sigma-1)\, \mathrm{P}_{i-1} \right] d\mu.$$

If $i + n$ be odd, or if $i > n$, the integrals on the right-hand side vanish. If $i < n$ and $i + n$ even, we must substitute,

$$\int_{-1}^{+1} \frac{d\mathrm{P}_n}{d\mu} \mathrm{P}_{i+1} = \int_{-1}^{+1} \frac{d\mathrm{P}_n}{d\mu} \mathrm{P}_{i-1} = 2, \text{ as found above,}$$

which gives

$$\int_{-1}^{+1} Q_n^{\sigma+2} Q_i^{\sigma} d\mu = 4 \frac{(i+\sigma)!}{(i-\sigma)!} (\sigma + 1).$$

When $n = i$, that part of the integral which depends on P_{i+1} vanishes, so that in that case

$$\int_{-1}^{+1} Q_n^{\sigma+2} Q_n^{\sigma} \, d\mu = - \frac{2}{2n+1} \frac{(n+\sigma)!}{(n-\sigma-2)!}.$$

More particularly when $\sigma = 0$ we have the integrals,

$$\int_{-1}^{+1} Q_n^2 P_i \, d\mu = 4, \text{ if } n + i \text{ is even and } i > n$$

$$= 0 \text{ in all other cases,}$$

$$\int_{-1}^{+1} Q_n^2 P_n \, d\mu = - 2 \frac{n \,.\, n-1}{2n+1}.$$

The result that $\int_{-1}^{+1} Q_n^{\sigma+2} Q_i^{\sigma} d\mu$ has zero value whenever $i > n$ can be extended to the more general integral $\int_{-1}^{+1} Q_n^{\sigma} Q_i^{\rho} \, d\mu$ provided then $\sigma > \rho$. To show this we need only consider the series,

$$Q_n^{\sigma} = A_0 Q_n^{\sigma-2} + A_2 Q_{n+2}^{\sigma-2} + \ldots,$$

where the coefficients may be determined from the integrals found above. Multiplying both sides with $Q_i^{\sigma-4}$ and integrating, all the terms vanish when $i > n$ and hence $\int Q_n^{\sigma} Q_i^{\sigma-4} = 0$ in that case. From $\sigma - 4$ we may proceed to $\sigma - 6$ and so on.

To obtain the integral $\int_{-1}^{+1} Q_n^{\sigma} Q_n^{\rho} d\mu$, we use formula (F), multiplying both sides by Q_n^{ρ} and integrating. We may without loss of generality take ρ to be smaller than σ. Utilising the result which has just been obtained, one integral on each side drops out and the equation becomes for even values of $\sigma + \rho$,

$$\int_{-1}^{+1} Q_n^{\sigma} Q_n^{\rho} = - (n - \sigma + 1)(n - \sigma + 2) \int_{-1}^{+1} Q_n^{\sigma-2} Q_n^{\rho} \, d\mu$$

$$= (n - \sigma + 1)(n - \sigma + 2)(n - \sigma + 3)(n - \sigma + 4) \int_{-1}^{+1} Q_n^{\sigma-4} Q_n^{\rho} d\mu$$

$$= (-1)^{\frac{\sigma-\rho}{2}} \frac{(n-\rho)!}{(n-\sigma)!} \int_{-1}^{+1} Q_n^{\sigma} Q_n^{\rho} \, d\mu$$

$$= (-1)^{\frac{\sigma-\rho}{2}} \frac{(n+\rho)!}{(n-\sigma)!} \cdot \frac{2}{2n+1} \text{ if } \rho < \sigma \text{ and } \rho + \sigma \text{ even}$$

$$= 0 \text{ if } \rho + \sigma \text{ is odd.}$$

Reverting now to equation F and multiplying both sides with Q^{ρ}_{n-2}, we obtain

$$\begin{aligned}
&\int_{-1}^{+1} Q_n^{\sigma} Q_{n-2}^{\rho} d\mu \\
&\quad = -(n-\sigma+1)(n-\sigma+2)\int_{-1}^{+1} Q_n^{\sigma-2} Q_{n-2}^{\rho} d\mu \\
&\qquad\qquad + (n+\sigma-2)(n+\sigma-3)\int_{-1}^{+1} Q_{n-2}^{\sigma-2} Q_{n-2}^{\rho} d\mu + \int_{-1}^{+1} Q_{n-2}^{\sigma} Q_{n-2}^{\rho} d\mu \\
&\quad = -(n-\sigma+1)(n-\sigma+2)\int_{-1}^{+1} Q_n^{\sigma-2} Q_{n-2}^{\rho} d\mu \\
&\qquad\qquad + (-1)^{\frac{\sigma-\rho+2}{2}} \frac{2}{2n-3}\left[(n+\sigma-2)(n+\sigma-3)\frac{(n-2+\rho)!}{(n-\sigma)!} - \frac{(n-2+\rho)!}{(n-\sigma-2)!}\right] \\
&\quad = -(n-\sigma+1)(n-\sigma+2)\int_{-1}^{+1} Q_n^{\sigma-2} Q_{n-2}^{\rho} d\mu - (-1)^{\frac{\sigma-\rho+2}{2}}\, 4(\sigma-1)\frac{(n+\rho-2)}{(n-\sigma)!}
\end{aligned}$$

The integral on the right-hand side may again be transformed in the same manner, changing from σ to $\sigma-2$. This leads to

$$\begin{aligned}
\int_{-1}^{+1} Q_n^{\sigma} Q_{n-2}^{\rho} d\mu &= (n-\sigma+1)(n-\sigma+2)(n-\sigma+3)(n-\sigma+4)\int_{-1}^{+1} Q_n^{\sigma-4} Q_{n-2}^{\rho} d\mu \\
&\qquad + (-1)^{\frac{\sigma-\rho+2}{2}}\, 4\,\frac{(n+\rho-2)!}{(n-\sigma)!}\{(\sigma-1)+(\sigma-3)\}.
\end{aligned}$$

If the same process be continued until the integral on the right-hand side becomes $\int_{-1}^{+1} Q_n^{\rho+2} Q_{n-2}^{\rho} d\mu$, the remaining terms on that side will consist of a series in arithmetical progression.

Adding this, we find

$$\begin{aligned}
\int_{-1}^{+1} Q_n^{\sigma} Q_{n-2}^{\rho} d\mu &= (-1)^{\frac{\sigma-\rho+2}{2}}\left[\frac{(n-2-\rho)!}{(n-\sigma)!}\int_{-1}^{+1} Q_n^{\rho+2} Q_{n-2}^{\rho} d\mu \right.\\
&\qquad\qquad \left. + \frac{(n-2+\rho)!}{(n-\sigma)!}(\sigma+\rho+2)(\sigma-\rho-2)\right] \\
&= (-1)^{\frac{\sigma-\rho+2}{2}}\left[\frac{(n-2-\rho)!}{(n-\sigma)!} \cdot \frac{(n-2+\rho)!}{(n-2-\rho)!}(4\rho+4)\right.\\
&\qquad\qquad \left. + \frac{(n-2+\rho)!}{(n-\sigma)!}(\sigma+\rho\, 2)(\sigma-\rho-2)\right] \\
&= (-1)^{\frac{\sigma-\rho+2}{2}}\frac{(n+\rho-2)!}{(n-\sigma)!}(\sigma^2-\rho^2), \quad \text{if } \sigma+\rho \text{ be even and } \rho<\sigma.
\end{aligned}$$

If $\sigma+\rho$ is odd or if $\rho>\sigma$ the integral is zero.

§ 7. $\int_{-1}^{+1} Q_n^\sigma P_i \, d\mu, \quad \int_{-1}^{+1} Q_n^\sigma Q_i^\rho \, d\mu.$

The integral of the product of a tesseral function and of a power of sin θ having been obtained in § 4, the above integrals are found by expanding the tesseral function or the zonal harmonic in a series proceeding by powers of sin θ.

From the expression of a tesseral function of degree n and type σ, as it is generally given, viz.:

$$\sin^\sigma \theta \left[A_0 \mu^{n-\sigma} + A_2 \mu^{n-\sigma-2} + \ldots \right],$$

it is seen by writing $x = \sin \theta$, $\mu = \sqrt{1 - x^2}$, and expanding the binomials, that the term of lowest power will be $\sin^\sigma \theta$, and that if $n - \sigma$ be even, the term of highest power is $\sin^n \theta$.

The coefficients of the series, which are given by HEINE, are most easily obtained by going back to the differential equation:

$$n(n+1) Q_n^\sigma + \frac{d}{d\mu} (1 - \mu^2) \frac{d}{d\mu} Q_n^\sigma = \frac{\sigma^2}{1 - \mu^2} Q_n^\sigma.$$

Substituting $x = \sqrt{1 - \mu^2}$, and changing variables, this becomes

$$n \,.\, n + 1 \,.\, Q_n^\sigma + \frac{1 - 2x^2}{x} \frac{dQ_n^\sigma}{dx} + (1 - x^2) \frac{d^2 Q_n^\sigma}{dx^2} = \frac{\sigma^2}{x^2} Q_n^\sigma.$$

If the series

$$a_\sigma x^\sigma + a_{\sigma+2} x^{\sigma+2} + \ldots a_q x^q + a_{q+2} x^{q+2} + \ldots$$

satisfies this differential equation, the coefficients a_{q+2} and a_q must be connected by the relation

$$a_q (n - q)(n + q + 1) + a_{q+2} (q - \sigma + 2)(q + \sigma + 2) = 0,$$

as is seen by substitution.

The first coefficient is determined by the fact that it is equal to the value of $d^\sigma P_n / d\mu^\sigma$ when $\mu = 1$. This quantity is known to be equal to $\frac{(n + \sigma)!}{(n - \sigma)!} \frac{1}{(2\sigma)!!}$. The other coefficients may now be determined in terms of this, and we find in this way for Q_n^σ the series

$$\frac{(n+\sigma)!}{(n-\sigma)!} \frac{x^\sigma}{(2\sigma)!!} \left[1 - \frac{(n-\sigma)(n+\sigma+1)}{1 \qquad \sigma+1} \left(\frac{x}{2}\right)^2 + \frac{(n-\sigma)(n-\sigma-2)(n+\sigma+1)(n+\sigma+3)}{1 \,.\, 2 \qquad\qquad \sigma+1 \,.\, \sigma+2} \left(\frac{x}{2}\right)^4 + \ldots \right].$$

The series breaks off with the term $x^{n-\sigma}$, when $n - \sigma$ is an even number, but also holds if $n - \sigma$ is odd.

The factor of x^λ in the series reduces to

$$(-1)^{\frac{1}{2}(\lambda-\sigma)} \frac{(n+\sigma)!!\,(n+\lambda-1)!!}{(n-\sigma-1)!!\,(n-\lambda)!!} \cdot \frac{1}{(\lambda-\sigma)!!\,(\lambda+\sigma)!!}.$$

When $(n-\sigma)$ is an odd number it will be more convenient to use a different series. Writing $Q = \mu N$, and changing the variable to $x = \sqrt{1-\mu^2}$ as before, the differential equation becomes

$$\overline{n-1}\,.\,\overline{n+2}\,N \frac{(1-4x^2)}{x} \frac{dN}{dx} + (1-x^2) \frac{d^2N}{dx^2} = \frac{\sigma^2 N}{x^3},$$

and from this we find for Q_n^σ the series

$$\frac{(n+\sigma)!\ \mu x^\sigma}{(n-\sigma)!\,(2\sigma)!!}\left[1 - \frac{(n-\sigma-1)(n+\sigma+2)}{1\,.\quad\sigma+1}\left(\frac{x}{2}\right)^2 + \frac{(n-\sigma-1)(n-\sigma-3)(n+\sigma+2)(n+\sigma+3)}{1\,.\,2\qquad\sigma+1\,.\,\sigma+2}\left(\frac{x}{2}\right)^4 - \dots\right].$$

The factor of μx^λ in the series is found to be

$$(-1)^{\frac{\lambda-\sigma}{2}} \frac{(n+\sigma-1)!!\,(n+\lambda)!!}{(n-\sigma)!!\,(n-\lambda-1)!!} \cdot \frac{1}{(\lambda-\sigma)!!\,(\lambda+\sigma)!!}.$$

In considering the integral $\int_{-1}^{+1} Q_n^\sigma Q_i^\rho \, d\mu$, we may, without loss of generality, take ρ to be smaller than σ. If of the two quantities $i-\rho$ and $n-\sigma$, one is odd and the other even, the integral vanishes.

If $(n-\sigma)$ and $(i-\rho)$ are both even numbers, we may express Q_n^σ in terms of a series of powers of $\sin\theta$ and obtain in this way :

$$\int_{-1}^{+1} Q_n^\sigma Q_i^\rho \, d\mu = \sum_{\lambda=\sigma}^{\lambda=n}{}^{(2)} (-1)^{\frac{\lambda-\sigma}{2}} \frac{(n+\sigma)!!\,(n+\lambda-1)!!}{(n-\sigma-1)!!\,(n-\lambda)!!\,(\lambda-\sigma)!!\,(\lambda+\sigma)!!} \int_{-1}^{+1} Q_i^\rho \sin^\lambda\theta \, d\mu$$

$$= c \sum_{\lambda=\sigma}^{\lambda=n}{}^{(2)} (-1)^{\frac{\lambda-\sigma}{2}} \frac{(n+\sigma)!!(n+\lambda-1)!!\,(i+\rho-1)!!(i-\lambda-2)!!(\rho+\lambda)!!}{(n-\sigma-1)!!(n-\lambda)!!\,(i-\rho)!!(i+\lambda+1)!!(\rho-\lambda-2)!!} \cdot \frac{1}{(\lambda-\sigma)!!(\lambda+\sigma)!!}.$$

The symbol $\sum\limits_{\lambda=\sigma}^{\lambda=n}{}^{(2)}$ is intended to express that λ takes successively the values σ, $\sigma+2, \dots n$, leaving out the alternate numbers. The constant c is equal to 2 or π, according as $\rho-\sigma$ is even or odd.

If $n-\sigma$ and $i-\rho$ are both odd, we find in the same way :

$$\int_{-1}^{+1} Q_n^\sigma Q_i^\rho \, d\mu = \sum_{\lambda=\sigma}^{\lambda=n-1}{}^{(2)} (-1)^{\frac{\lambda-\sigma}{2}} \frac{(n+\sigma-1)!!\,(n+\lambda)!!}{(n-\sigma)!!\,(n-\lambda-1)!!\,(\lambda-\sigma)!!\,(\lambda+\sigma)!!} \int_{-1}^{+1} \mu Q_i^\rho \sin^\lambda\theta \, d\mu$$

$$= c \sum_{\lambda=\sigma}^{\lambda=n-1}{}^{(2)} (-1)^{\frac{\lambda-\sigma}{2}} \frac{(n+\sigma-1)!!(n+\lambda)!!\,(i+\rho)!!(i-\lambda-3)!!(\rho+\lambda)!!}{(n-\sigma)!!(n-\lambda-1)!!(i-\rho-1)!!(i+\lambda+2)!!(\rho-\lambda-2)!!} \cdot \frac{1}{(\lambda-\sigma)!!(\lambda+\sigma)!!}.$$

The value of c is the same as before.

Our result shows that when $\rho-\sigma$ is even, the integral vanishes when $i > n$. For when $\rho-\sigma$ is even, $\rho-\lambda-2$ will also be even, for all values of λ, and it will be negative because σ is the smallest value of λ and σ is larger than ρ. $1/(\rho - \lambda - 2)\,!\,!$ must therefore be zero, and all the terms of the sum vanish unless there is a negative even factorial in the numerator. The only factorial which can become negative is $(i - \lambda - 2)\,!\,!$, but n being the highest value of λ, it will not be negative when $i > n$. For the special case, $i = n$, all the terms of the sum vanish except the last, for which $\lambda = n$. We have in that case

$$\frac{(i - \lambda - 2)\,!\,!}{(\rho - \lambda - 2)\,!\,!} = \frac{(i - n - 2)\,!\,!}{(\rho - n - 2)\,!\,!} = \frac{(-2)\,!\,!}{(\rho - n - 2)\,!\,!} = (-1)^{\frac{n-\rho}{2}} (n - \rho)\,!\,!,$$

and the integral reduces to $(-1)^{\frac{\sigma-\rho}{2}} \frac{2}{2n+1} \frac{(n+\rho)\,!}{(n-\sigma)\,!}$, as already found in § 6.

The same expression is found when $n - \sigma$ is odd.

Another and generally more convenient expression for $\int_{-1}^{+1} Q_n^\sigma Q_i^\rho \, d\mu$ is found by expanding Q_i^ρ instead of Q_n^σ in terms of powers of $\sin\theta$. We thus obtain, when $n - \sigma$ and $i - \rho$ are both even,

$$\int_{-1}^{+1} Q_n^\sigma Q_i^\rho \, d\mu = c \sum_{\lambda=\rho}^{\lambda=i}{}^{(2)} (-1)^{\frac{\lambda-\rho}{2}} \frac{(n+\sigma-1)!!(i+\rho)!!(i+\lambda-1)!!(n-\lambda-2)!!}{(n-\sigma)\,!!\,(i-\rho-1)\,!!\,(i-\lambda)\,!!\,(n+\lambda+1)\,!!} \frac{1}{(\lambda-\rho)!!(\lambda+\rho)!!} \frac{(\sigma+\lambda)!!}{(\sigma-\lambda-2)!!},$$

and when $n - \sigma$ and $i - \rho$ are both odd,

$$= c \sum_{\lambda=\rho}^{\lambda=i-1}{}^{(2)} (-1)^{\frac{\lambda-\rho}{2}} \frac{(n+\sigma)\,!!\,(i+\rho-1)\,!!\,(n-\lambda-3)\,!!\,(i+\lambda)\,!!}{(n-\sigma-1)!!(i-\rho)!!(n+\lambda+2)!!(i-\lambda-1)\,!!} \frac{1}{(\lambda-\rho)!!(\lambda+\rho)!!} \frac{(\sigma+\lambda)!!}{(\sigma-\lambda-2)!!}.$$

Writing

$$A_0 = \frac{(\sigma+\rho)(\sigma+\rho-2)\ldots(\sigma-\rho+2)(\sigma-\rho)}{(2\rho)\,!\,!},$$

$$A_2 = \frac{(\sigma+\rho+2)(\sigma+\rho)\ldots(\sigma-\rho)(\sigma-\rho-2)}{2\,!\,!\,(2\rho+2)\,!\,!},$$

$$A_4 = \frac{(\sigma+\rho+4)(\sigma+\rho+2)\ldots(\sigma-\rho-2)(\sigma-\rho-4)}{4\,!\,!\,(2\rho+4)\,!\,!},$$

we may put the integral into the following form :

$\int_{-1}^{+1} Q_n^\sigma Q_i^\rho \, d\mu$, if $(n - \sigma)$ even, $(i - \rho)$ even, and excluding $n = i$

$$= c \frac{(i+\rho)\,!}{(i-\rho)\,!} \frac{(n+\sigma-1)\,!\,!}{(n-\sigma)\,!\,!} \frac{(n-\rho-2)\,!\,!}{(n+\rho+1)\,!\,!} \Sigma,$$

where

$$\Sigma = A_0 - A_2 \frac{(i+\rho+1)(i-\rho)}{(n-\rho-2)(n+\rho+3)} + A_4 \frac{(i+\rho+1)(i+\rho+3)(i-\rho)(i-\rho-2)}{(n-\rho-2)(n-\rho-4)(n+\rho+3)(n+\rho+5)} \ldots$$

where $c = 2$, if $\sigma - \rho$ is even, and $c = \pi$, if $\sigma - \rho$ is odd, the last term of the series is

$$\mathrm{A}_{i-\rho}\frac{[(i+\rho+1)(i+\rho+3)\ldots(2i-1)]\,[(i-\rho)(i-\rho-2)\ldots 2]}{[(n-\rho-2)(n-\rho-4)\ldots(n-i)]\,[(n+\rho+3)(n+\rho+5)\ldots(n+i+1)]},$$

but the series breaks off before the end, if $\sigma-\rho$ is even and $i>\sigma$ owing to the factor A vanishing. The number of terms in that case is $\sigma-\rho-1$.

Similarly when $(n-\sigma)$ and $(i-\rho)$ are both odd

$$\int_{-1}^{+1}\mathrm{Q}_n^\sigma\mathrm{Q}_i^\sigma\,d\mu = c\,\frac{(i+\rho)!}{(i-\rho)!}\,\frac{(n+\sigma)!!\,(n-\rho-3)!!}{(n-\sigma-1)!!\,(n+\rho+2)!!}\,\Sigma,$$

where

$$\Sigma = \mathrm{A}_0 = \mathrm{A}_2\frac{(i+\rho+2)(i-\rho-1)}{(n-\rho-3)(n+\rho+4)} + \mathrm{A}_4\frac{(i+\rho+2)(i+\rho+4)(i-\rho-1)(i-\rho-3)}{(n-\rho-3)(n-\rho-5)(n+\rho+4)(n+\rho+6)}\cdots$$

where c and the coefficients A have the same value as before. The last term of the series is

$$\mathrm{A}_{i-\rho-1}\frac{[(i+\rho+2)(i+\rho+4)\ldots(2i-1)]\,[(i+\rho+1)(i-\rho-3)\ldots 2]}{[(n-\rho-3)(n-\rho-5)\ldots(n-i)]\,[(n+\rho+4)(n+\rho+6)\ldots(n+i+1)]}.$$

To obtain $\int\mathrm{Q}_n^\sigma\mathrm{P}_i\,d\mu$, we need only put $\rho=0$ in the previous investigation. This gives:

$$\mathrm{A}_0=\sigma,\qquad \mathrm{A}_2=\frac{\sigma+2\,.\,\sigma\,.\,\sigma-2}{2\,.\,2},\qquad \mathrm{A}_4=\frac{\sigma+4\,.\,\sigma+2\,.\,\sigma\,.\,\sigma-2\,.\,\sigma-4}{2\,.\,4\,.\,2\,.\,4},$$

$$\int_{-1}^{+1}\mathrm{Q}_n^\sigma\mathrm{P}_i\,d\mu = 2\,\frac{(n+\sigma-1)!!\,(n-2)!!}{(n-\sigma)!!\,(n+1)!!}\left[\mathrm{A}_0-\mathrm{A}_2\frac{i\,.\,i+1}{n-2\,.\,n+3}+\mathrm{A}_4\frac{i.i-2.i+1.i+3}{n-2.n-4.n+3.n+5}\cdots\right],$$

if i and σ is even, and

$$\int_{-1}^{+1}\mathrm{Q}_n^\sigma\mathrm{P}_i\,d\mu = 2\,\frac{(n+\sigma)!!\,(n-3)!!}{(n-\sigma-1)!!\,(n+2)!!}\left[\mathrm{A}_0-\mathrm{A}_2\frac{i-1\,.\,i+2}{n-3\ n+4}+\mathrm{A}_4\frac{i-1.i-3\ i+2.i+4}{n-3.n-5\ n+4.n+6}\cdots\right],$$

if σ is even and i odd. Thus, if i be even

$$\int_{-1}^{+1}\mathrm{Q}_n{}^2\mathrm{P}_i\,d\mu = 4,$$

$$\int_{-1}^{+1}\mathrm{Q}_n{}^4\mathrm{P}_i\,d\mu = 8\,[(n+3\,.\,n-2)-3\,(i\,.\,i+1)],$$

$$\int_{-1}^{+1}\mathrm{Q}_n{}^6\mathrm{P}_i\,d\mu = 12\,[(n+5\,.\,n+3\,.\,n-2\,.\,n-4)-8\,(i\,.\,i+1\,.\,n+5\,.\,n-4)$$
$$+\,10\,(i\,.\,i-2\,.\,i+1\,.\,i+3)],$$

and if i be odd:

$$\int_{-1}^{+1} \mathrm{Q}_n{}^2 \mathrm{P}_i \, d\mu = 4,$$

$$\int_{-1}^{+1} \mathrm{Q}_n{}^4 \mathrm{P}_i \, d\mu = 8 \left[(n + 4 \,.\, n - 3) - 3 \, (i + 2 \,.\, i - 1)\right],$$

$$\int_{-1}^{+1} \mathrm{Q}_n{}^6 \mathrm{P}_i \, d\mu = 12 \left[(n + 6 \,.\, n + 4 \,.\, n - 3 \,.\, n - 5) - 8 \, (i + 2 \,.\, i - 1 \,.\, n + 6 \,.\, n - 5) + 10 \, (i + 2 \,.\, i + 4 \,.\, i - 1 \,.\, i - 3)\right].$$

The equations do not hold for $i = n$ as has already been explained.

§ 8. $\int_{-1}^{+1} \mathrm{Q}_n^\sigma \sin p\theta \, d\mu$, $\int_{-1}^{+1} \mathrm{Q}_n^\sigma \cos p\theta \, d\mu$.

These important integrals are obtained in two different forms according as Q_n^σ or the trigonometrical functions are expressed in terms of the powers of $\sin \theta$. In the former case the problem reduces itself to the evaluation of integrals of the form

$$\int_0^\pi \sin^\lambda \theta \sin p\theta \, d\rho \quad \text{and} \quad \int_0^\pi \sin^\lambda \theta \cos \theta \sin p\theta \, d\theta.$$

It may easily be proved that

$$\int_{-1}^{+1} \sin^\lambda \theta \sin p\theta \, d\mu = (-1)^{\frac{p-1}{2}} c \frac{(\lambda + 1)\,!}{(\lambda + 1 + p)\,!\,!\,(\lambda + 1 - p)\,!\,!} \text{ when } p \text{ is odd,}$$

$$= 0 \text{ when } p \text{ is even.}$$

$$\int_{-1}^{+1} \sin^\lambda \theta \cos p\theta \, d\mu = (-1)^{\frac{p}{2}} c \frac{(\lambda + 1)\,!}{(\lambda + 1 + p)\,!\,!\,(\lambda + 1 - p)\,!\,!} \text{ when } p \text{ is even,}$$

$$= 0 \text{ when } p \text{ is odd.}$$

$$\int_{-1}^{+1} \sin^{\lambda-1} \theta \cos \theta \sin p\theta \, d\mu = (-1)^{\frac{p+2}{2}} cp \frac{\lambda\,!}{(\lambda + 1 + p)\,!\,!\,(\lambda + 1 - p)\,!\,!} \text{ when } p \text{ is even,}$$

$$= 0 \text{ when } p \text{ is odd.}$$

$$\int_{-1}^{+1} \sin^{\lambda-1} \theta \cos \theta \cos p\theta \, d\mu = (-1)^{\frac{p-1}{2}} cp \frac{\lambda\,!}{(\lambda + 1 + p)\,!\,!\,(\lambda + 1 - p)\,!\,!} \text{ when } p \text{ is odd,}$$

$$= 0 \text{ when } p \text{ is even.}$$

In these equations c is equal to 2 or π according as $(p + \lambda)$ is even or odd.

From the results of § 7 we may now write, if $n - \sigma$ be even and p odd,

$$\int_{-1}^{+1} \mathrm{Q}_n^\sigma \sin p\theta \, d\mu$$

$$= \sum_{\lambda=\sigma}^{\lambda=n}{}^{(2)} (-1)^{\frac{\lambda-\sigma}{2}} \frac{(n+\sigma)!!(n+\lambda-1)!!}{(n-\sigma-1)!!(n-\lambda)!!} \frac{1}{(\lambda-\sigma)!!(\lambda+\sigma)!!} \int_{-1}^{+1} \sin^\lambda\theta \sin p\theta \, d\mu$$

$$= c \sum_{\lambda=\sigma}^{\lambda=n}{}^{(2)} (-1)^{\frac{\lambda+p-\sigma-1}{2}} \frac{(n+\sigma)!!(n+\lambda-1)!!}{(n-\sigma-1)!!(n-\lambda)!!} \frac{(\lambda+1)!}{(\lambda-\sigma)!!(\lambda+\sigma)!!} \cdot \frac{1}{(\lambda+1+p)!!(\lambda+1-p)!!} .$$

When p and $n - \sigma$ are both even the integral vanishes.

If $(n - \sigma)$ be odd and p even,

$$\int_{-1}^{+1} \mathrm{Q}_n^\sigma \sin p\theta \, d\mu$$

$$= \sum_{\lambda=\sigma}^{\lambda=n-1}{}^{(2)} (-1)^{\frac{\lambda-\sigma}{2}} \frac{(n+\sigma-1)!!(n+\lambda)!!}{(n-\sigma)!!(n-\lambda-1)!!} \frac{1}{(\lambda-\sigma)!!(\lambda+\sigma)!!} \int_{-1}^{+1} \cos\theta \sin^\lambda\theta \sin p\theta \, d\mu$$

$$= cp \sum_{\lambda=\sigma}^{\lambda=n-1}{}^{(2)} (-1)^{\frac{\lambda+p-\sigma+2}{2}} \frac{(n+\sigma-1)!!(n+\lambda)!!}{(n-\sigma)!!(n-\lambda-1)!!} \frac{(\lambda+1)!}{(\lambda-\sigma)!!(\lambda+\sigma)!!} \frac{1}{(\lambda+2+p)!!(\lambda+2-p)!!} .$$

It will be noticed that the integral is zero whenever $(n + \sigma + p)$ is even. Hence if $\sin p\theta$ is expressed in terms of tesseral functions, p being even, only odd values of n occur when σ is even, and only even values when σ is an odd number. The reverse will be the case when p is odd.

We find similarly, if $(n - \sigma)$ is odd and p odd,

$$\int_{-1}^{+1} \mathrm{Q}_n^\sigma \cos p\theta \, d\mu$$

$$= \sum_{\lambda=\sigma}^{\lambda=n-1}{}^{(2)} (-1)^{\frac{\lambda-\sigma}{2}} \frac{(n+\sigma-1)!!(n+\lambda)!!}{(n-\sigma)!!(n-\lambda-1)!!} \frac{1}{(\lambda-\sigma)!!(\lambda+\sigma)!!} \int_{-1}^{+1} \cos\theta \sin^\lambda\theta \cos p\theta \, d\mu$$

$$= cp \sum_{\lambda=\sigma}^{\lambda=n-1}{}^{(2)} (-1)^{\frac{\lambda+p-\sigma-1}{2}} \frac{(n+\sigma-1)!!(n+\lambda)!!}{(n-\sigma)!!(n-\lambda-1)!!} \frac{(\lambda+1)!}{(\lambda-\sigma)!!(\lambda+\sigma)!!} \frac{1}{(\lambda+2+p)!!(\lambda+2-p)!!} ,$$

and when $n - \sigma$ and p are both even,

$$\int_{-1}^{+1} \mathrm{Q}_n^\sigma \cos p\theta \, d\mu$$

$$= c \sum_{\lambda=\sigma}^{\lambda=n}{}^{(2)} (-1)^{\frac{\lambda-\sigma+p}{2}} \frac{(n+\sigma)!!(n+\lambda-1)!!}{(n-\sigma-1)!!(n-\lambda)!!} \frac{(\lambda+1)!}{(\lambda-\sigma)!!(\lambda+\sigma)!!} \frac{1}{(\lambda+1+p)!!(\lambda+1-p)!!} ,$$

when $n - \sigma + p$ is odd the integrals vanish for all values of p.

We may now prove the theorem which has been mentioned in the first paragraph. If σ and p are both even, $\int_{-1}^{+1} \mathrm{Q}_n^\sigma \sin p\theta \, d\mu$ is zero, unless n is odd. In that case the quantity $(\lambda + 2 - p)!!$ which occurs in the denominator will be even, as λ takes up successively the values σ, $\sigma + 2$, &c. The highest value which $\lambda + 2 - p$ can take is $n + 1 - p$, and this is negative when $p > n + 1$. It follows that

$$\int_{-1}^{+1} \mathrm{Q}_n^\sigma \sin p\theta \, d\mu = 0 \quad \text{when} \quad p > n + 1 \text{ and } \sigma \text{ is even.}$$

The restriction that p is even is not necessary, because when p is odd n will be even and the factorial in the denominator, according to the above equations, is $(\lambda + 1 - p)!!$ This again will be infinitely large whenever it is negative, or taking the highest value of λ, which is now n, whenever $p > n + 1$.

We prove exactly in the same way that

$$\int_{-1}^{+1} \mathrm{Q}_n^\sigma \cos p\theta \, d\mu = 0 \text{ when } p > n + 1 \text{ and } \sigma \text{ is odd.}$$

The value of the integrals in the general case may conveniently be expressed by two series Σ_1 and Σ_2 defined by

$$\frac{(n - p + 1)!!\,(n + p + 1)!!\,(n - \sigma)!}{(2n - 1)!!\,(n + 1)!}\,\Sigma_1 \equiv 1 - \frac{(n + \sigma)(n - \sigma)(n + p + 1)(n - p + 1)}{2 \,.\, n + 1 \,.\, n \,.\, 2n - 1}$$
$$+ \frac{(n + \sigma)(n + \sigma - 2)(n - \sigma)(n - \sigma - 2)(n + p + 1)(n + p - 1)(n - p + 1)(n - p - 1)}{2 \,.\, 4 \qquad n + 1 \,.\, n \,.\, n - 1 \,.\, n - 2 \qquad 2n - 1 \,.\, 2n - 3} \dots$$

$$\frac{(n - p + 1)!!\,(n + p + 1)!!\,(n - \sigma)!}{(2n - 1)!!\,n!\,p}\,\Sigma_1 \equiv 1 - \frac{(n + \sigma - 1)(n - \sigma - 1)(n + p + 1)(n - p + 1)}{2 \,.\, n \,.\, n - 1 \,.\, 2n - 1}$$
$$+ \frac{(n + \sigma - 1)(n + \sigma - 3)(n - \sigma - 1)(n - \sigma - 3)(n + p + 1)(n + p - 1)(n - p + 1)(n - p - 1)}{2 \,.\, 4 \qquad n \,.\, n - 1 \,.\, n - 2 \,.\, n - 3 \qquad 2n - 1 \;\, 2n - 3} \dots$$

Then

$$\int_{-1}^{+1} \mathrm{Q}_n^\sigma \sin p\theta \, d\mu = (-1)^{\frac{n - \sigma + p - 1}{2}} c\,\Sigma_1 \text{ if } p \text{ be odd and } n + \sigma \text{ is even.}$$
$$= (-1)^{\frac{n - \sigma + p + 1}{2}} c\,\Sigma_2 \text{ if } p \text{ be even and } n + \sigma \text{ is odd.}$$
$$= 0 \text{ whenever } n + p + \sigma \text{ is even.}$$

$$\int_{-1}^{+1} \mathrm{Q}_n^\sigma \cos p\theta \, d\mu = (-1)^{\frac{n - \sigma + p}{2}} c\,\Sigma_1 \text{ if } p \text{ be even and } n + \sigma \text{ is even.}$$
$$= (-1)^{\frac{n - \sigma + p + 2}{2}} c\,\Sigma_2 \text{ if } p \text{ be odd and } n + \sigma \text{ is odd.}$$
$$= 0 \text{ whenever } n + p + \sigma \text{ is odd.}$$

The constant c takes the value 2 or π, according as $p + n$ is even or odd. The series Σ_1 and Σ_2 break off in all cases ultimately when one of the factors of the products $(n = \sigma)$. $(n - \sigma - 2) \dots$ or $(n - \sigma - 1)$ $(n - \sigma - 3) \dots$ become zero. When $n - p$ is odd, the series may break off before it has reached its full number of terms, owing to one of the factors in the product $(n-p+1)(n-p-1)\ldots$ becoming zero. In the special case $n=\sigma$ the value of Σ_1 reduces to

$$\frac{(2n-1)!!\,(n+1)!}{(n-p+1)!!\,(n+p+1)!!}.$$

The above equations are inconvenient for numerical calculations, unless $n-\sigma$ be small, or $n-p$ be even and small. We obtain quite different and generally more convenient expressions, if we begin by expressing $\sin p\theta$ or $\cos p\theta$ in terms of a series proceeding by powers of $\sin\theta$.

Let

$$B_0 = 1\,;\ B_1 = p\,;\ B_2 = \frac{p-1\,p+1}{1\,.\,2}\,;\ B_3 = \frac{p-2\,.\,p\,.\,p+2}{1\,.\,2\,.\,3}\,;$$

and generally

$$B^{\lambda} = \frac{(p-\lambda+1)(p-\lambda+3)\ldots(p+\lambda-3)(p+\lambda-1)}{\lambda\,!}$$

$$= \frac{(p+\lambda-1)!!}{(p-\lambda-1)!!}\,\frac{1}{\lambda\,!}.$$

Also put

$$C_0 = 1\,;\ C_1 = p\,.\,;\ C_2 = p\,.\,\frac{p}{2}\,;\ C_3 = p\,.\,\frac{p-1\,.\,p+1}{1\,.\,2\,.\,3}\,;\ C_4 = p\,.\,\frac{p-2\,.\,p\,.\,p+2}{4\,!}.$$

$$C_{\lambda} = p\,.\,\frac{(p-\lambda+2)(p-\lambda+4)\ldots(p+\lambda-4)(p+\lambda-2)}{\lambda\,!}$$

$$= \frac{p}{\lambda\,!}\,.\,\frac{(p+\lambda-2)!!}{(p-\lambda)!!}$$

Then the well-known expressions for the trigonometrical functions of the multiples of an angle may be written :

If p be even :

$$\frac{\sin p\theta}{\cos\theta} = B_1\sin\theta - B_3\sin^3\theta + B_5\sin^5\theta - \ldots \pm B_{p-1}\sin^{p-1}\theta,$$

$$\cos p\theta = C_0 - C_2\sin^2\theta + C_4\sin^4\theta \qquad - \ldots \pm C_p\sin^p\theta\,;$$

and if p be odd :

$$\sin p\theta = C_1\sin\theta - C_3\sin^3\theta + C_5\sin^5\theta \quad - \ldots \pm C_p\sin^p\theta,$$

$$\frac{\cos p\theta}{\cos\theta} = B_0 - B_2\sin^2\theta + B_4\sin^4\theta \qquad - \ldots \pm B_{p-1}\sin^{p-1}\theta.$$

We derive from this, p being even :

$$\int_{-1}^{+1} Q_n^{\sigma}\sin p\theta\,d\mu = \sum_{\lambda=1}^{\lambda=p-1}{}^{(2)}\,(-1)^{\frac{\lambda-1}{2}}\,B_{\lambda}\int_{-1}^{+1} Q_n^{\sigma}\sin^{\lambda}\theta\cos\theta\,d\mu,$$

$$\int_{-1}^{+1} Q_n^{\sigma}\cos p\theta\,d\mu = \sum_{\lambda=0}^{\lambda=p}{}^{(2)}\,(-1)^{\frac{\lambda}{2}}\,C_{\lambda}\int_{-1}^{+1} Q_n^{\sigma}\sin^{\lambda}\theta\,d\mu,$$

and p being odd:

$$\int_{-1}^{+1} \mathrm{Q}_n^\sigma \sin p\theta \, d\mu = \sum_{\lambda=1}^{\lambda=p} {}^{(2)} (-1)^{\frac{\lambda-1}{2}} \mathrm{C}_\lambda \int_{-1}^{+1} \mathrm{Q}_n^\sigma \sin^\lambda \theta \, d\mu,$$

$$\int_{-1}^{+1} \mathrm{Q}_n^\sigma \cos p\theta \, d\mu = \sum_{\lambda=0}^{\lambda=p-1} {}^{(2)} (-1)^{\frac{\lambda}{2}} \mathrm{B}_\lambda \int_{-1}^{+1} \mathrm{Q}_n^\sigma \sin^\lambda \theta \cos \theta \, d\mu.$$

Hence from the results of § 4, if p be even and $(n - \sigma)$ odd:

$$\int_{-1}^{+1} \mathrm{Q}_n^\sigma \sin p\theta \, d\mu = c \sum_{\lambda=1}^{\lambda=p-1} {}^{(2)} (-1)^{\frac{\lambda-1}{2}} \mathrm{B}^\lambda \frac{(n+\sigma)!!\,(n-\lambda-3)!!\,(\sigma+\lambda)!}{(n-\sigma-1)!!\,(n+\lambda+2)!!\,(\sigma-\lambda-2)!!}.$$

The constant c is 2 or π, according as σ is odd or even.

If p be odd and $n - \sigma$ even, and with the same meaning of c:

$$\int_{-1}^{+1} \mathrm{Q}_n^\sigma \sin p\theta \, d\mu = c \sum_{\lambda=1}^{\lambda=p} {}^{(2)} (-1)^{\frac{\lambda-1}{2}} \mathrm{C}_\lambda \frac{(n+\sigma-1)!!\,(n-\lambda-2)!!\,(\sigma+\lambda)!!}{(n-\sigma)!!\,(n+\lambda+1)!!\,(\sigma-\lambda-2)!!}.$$

Similarly if p and $n - \sigma$ are both even,

$$\int_{-1}^{+1} \mathrm{Q}_n^\sigma \cos p\theta \, d\mu = c \sum_{\lambda=0}^{\lambda=p} {}^{(2)} (-1)^{\frac{\lambda}{2}} \mathrm{C}_\lambda \frac{(n+\sigma-1)!!\,(n-\lambda-2)!!\,(\sigma+\lambda)!!}{(n-\sigma)!!\,(n+\lambda+1)!!\,(\sigma-\lambda-2)!!},$$

and if p and $n - \sigma$ be both odd,

$$\int_{-1}^{+1} \mathrm{Q}_n^\sigma \cos p\theta \, d\mu = c \sum_{\lambda=0}^{\lambda=p-1} {}^{(2)} (-1)^{\frac{\lambda}{2}} \mathrm{B}_\lambda \frac{(n+\sigma)!!\,(n-\lambda-3)!!\,(\sigma+\lambda)!!}{(n-\sigma-1)!!\,(n+\lambda+2)!!\,(\sigma-\lambda-2)!!}.$$

In the two last equations c is equal to 2 when σ is even and equal to π when σ is odd. In the interpretation of the summation a little care is required when $\sigma - \lambda - 2$ and $n - \lambda - 2$ or $n - \lambda - 3$ are negative, as the alternate factorials are infinite for even values of the argument.

To put the equations into a form useful for numerical calculations, it is convenient to designate by separate letters the following six series:—

$$\mathrm{M}_n^\sigma \equiv \mathrm{C}_1 \,.\, \sigma - 1 \,.\, \sigma + 1 - \mathrm{C}_3 \frac{\sigma-3 \,.\, \sigma-1 \,.\, \sigma+1 \,.\, \sigma+3}{n-3 \,.\, n+4} + \mathrm{C}_5 \frac{\sigma-5 \,.\, \sigma-3 \,.\, \sigma-1 \,.\, \sigma+1 \,.\, \sigma+3 \,.\, \sigma+5}{n-3 \,.\, n-5 \,.\, n+4 \,.\, n+6} - \ldots$$

$$\mathrm{S}_n^\sigma \equiv \mathrm{B}_1 \,.\, \sigma - 1 \,.\, \sigma + 1 - \mathrm{B}_3 \frac{\sigma-3 \,.\, \sigma-1 \,.\, \sigma+1 \,.\, \sigma+3}{n-4 \,.\, n+5} + \mathrm{B}_5 \frac{\sigma-5 \,.\, \sigma-3 \,.\, \sigma-1 \,.\, \sigma+1 \,.\, \sigma+3 \,.\, \sigma+5}{n-4 \,.\, n-6 \,.\, n+5 \,.\, n+7} - \ldots$$

$$\mathrm{U}_n^\sigma \equiv \mathrm{C}_0 \,.\, \sigma - \mathrm{C}_2 \frac{\sigma-2 \,.\, \sigma \,.\, \sigma+2}{n-2 \,.\, n+3} + \mathrm{C}_4 \frac{\sigma-4 \,.\, \sigma-2 \,.\, \sigma \,.\, \sigma+2 \,.\, \sigma+4}{n-2 \,.\, n-4 \,.\, n+3 \,.\, n+5} - \ldots$$

$$V_n^\sigma \equiv B_0\sigma - B_2\frac{\sigma-2\,.\,\sigma\,.\,\sigma+2}{n-3\,.\,n+4} + B_4\frac{\sigma-4\,.\,\sigma-2\,.\,\sigma\,.\,\sigma+2\,.\,\sigma+4}{n-3\,.\,n-5\,.\,n+4\,.\,n+6} - \ldots$$

$$K_n^\sigma \equiv C_n - C_{n+2}\frac{n+2-\sigma\,.\,n+2+\sigma}{2\,.\,2n+3}$$

$$+ C_{n+4}\frac{n+2-\sigma\,.\,n+4-\sigma\,.\,n+2+\sigma\,.\,n+4+\sigma}{2\,.\,4\quad 2n+3\,.\,2n+5} - \ldots$$

$$N_n^\sigma \equiv B_{n-1} - B_{n+1}\frac{n+1-\sigma\,.\,n+1+\sigma}{2\,.\,2n+3}$$

$$B_{n+3}\frac{n+1-\sigma\,.\,n+3-\sigma\,.\,n+1+\sigma\,.\,n+3+\sigma}{2\,.\,4\quad 2n+3\,.\,2n+5} - \ldots$$

The series are all continued until they break off, which may happen either because the factors B, C, or one of the other factors takes zero value.

With the help of the functions just defined, the integrals now take the form

$$\int_{-1}^{+1} Q_n^\sigma \sin p\theta\, d\mu = \pi\frac{(n+\sigma-1)!!\,(n-3)!!}{(n-\sigma)!!\,(n+2)!!}M_n^\sigma \text{ if } \sigma \text{ be even, } p \text{ odd, } n \text{ even.}$$

$$= \pi\frac{(n+\sigma)!!\,(n-4)!!}{(n-\sigma-1)!!\,(n+3)!!}S_n^\sigma \text{ if } \sigma \text{ be even, } p \text{ even, } n \text{ odd.}$$

$$= 2\left[\frac{(n+\sigma-1)!!\,(n-3)!!}{(n-\sigma)!!\,(n+2)!!}M_n^\sigma + (-1)^{\frac{\sigma-1}{2}}\frac{(n+\sigma)!}{(2n+1)!!}K_n^\sigma\right]$$

$$\text{if } \sigma \text{ be odd, } p \text{ odd, } n \text{ odd.}$$

$$= 2\left[\frac{(n+\sigma)!!\,(n-4)!!}{(n-\sigma\quad 1)!!\,(n+3)!!}S_n^\sigma + (=1)^{\frac{\sigma-1}{2}}\frac{(n+\sigma)!}{(2n+1)!!}N_n^\sigma\right]$$

$$\text{if } \sigma \text{ be odd, } p \text{ even, } n \text{ even.}$$

$$= 0 \text{ if } n+\sigma+p \text{ be even.}$$

$$\int_{-1}^{+1} Q_n^\sigma \cos p\theta\, d\mu = 2\left[\frac{(n+\sigma-1)!!(n-2)!!}{(n-\sigma)!!\,(n-1)!!}U_n^\sigma + (-1)^{\frac{\sigma}{2}}\frac{(n+\sigma)!}{(2n+1)!!}K_n^\sigma\right]$$

$$\text{if } \sigma \text{ be even, } p \text{ even, } n \text{ even.}$$

$$= 2\left[\frac{(n+\sigma)!!\,(n-3)!!}{(n-\sigma-1)!!\,(n+2)!!}V_n^\sigma + (-1)^{\frac{\sigma}{2}}\frac{(n+\sigma)!}{(2n+1)!!}N_n^\sigma\right]$$

$$\text{if } \sigma \text{ be even, } p \text{ odd, } n \text{ odd.}$$

$$= \pi\frac{(n+\sigma-1)!!\,(n-2)!!}{(n-\sigma)!!\,(n+1)!!}U_n^\sigma \text{ if } \sigma \text{ be odd, } p \text{ even, } n \text{ odd.}$$

$$= \pi\frac{(n+\sigma)!!\,(n-3)!!}{(n-\sigma-1)!!\,(n+2)!!}V_n^\sigma \text{ if } \sigma \text{ be odd, } p \text{ odd, } n \text{ even.}$$

$$= 0 \text{ if } n+\sigma+p \text{ be odd.}$$

By comparing these last equations with the results obtained when the same integrals were expressed in terms of the series Σ_1 and Σ_2, we may derive the following special values:

$$\begin{array}{lllllll}
M^\sigma_n = 0 & \text{if } \sigma \text{ be even,} & p \text{ odd,} & n \text{ even} & \text{and} & p \geqq n + 3, \\
S^\sigma_n = 0 & ,, \quad \sigma \quad ,, \quad ,, & p \text{ even,} & n \text{ odd} & ,, & p \geqq n + 3, \\
U^\sigma_n = 0 & ,, \quad \sigma \quad ,, \quad \text{odd,} & p \quad ,, & n \quad ,, & ,, & p \geqq n + 3, \\
V^\sigma_n = 0 & ,, \quad \sigma \quad ,, \quad ,, & p \text{ odd,} & n \text{ even} & ,, & p \geqq n + 3.
\end{array}$$

The two series marked N^σ_n and K^σ_n had to be introduced because the series for M^σ_n, S^σ_n, U^σ_n, V^σ_n break off as soon as one of the factors becomes zero. But in the original summation which gave rise to, *e.g.*, M^σ_n, viz.:

$$\overset{\lambda=p}{\underset{\lambda=1}{\Sigma^2}} C_\lambda \frac{(n+\sigma-1)!!\,(n-\lambda-2)!!\,(\sigma+\lambda)!!}{(n-\sigma)!!\,(n+\lambda+1)!!\,(\sigma-\lambda-2)!!},$$

$(\sigma - \lambda - 2)!!$ begins to be infinite when $\lambda = \sigma$, and hence the terms of the series will drop out until $(n - \lambda - 2)$ is negative, *i.e.*, until $\lambda \geqq n$. For higher values of λ, $\frac{(n-\lambda-2)!!}{(\sigma-\lambda-2)!!}$ will be finite again. There is, therefore, a second portion of the series not included under M^σ_n, and it is this second portion which appears as K^σ_n.

§ 9. *Relations between* M^σ_n, S^σ_n, U^σ_n, V^σ_n.

Certain relations exist between the series which serve to express the integrals $\int_{-1}^{+1} Q^\sigma_n \cos p\theta \, d\mu$ and $\int_{-1}^{+1} Q^\sigma_n \sin p\theta \, d\mu$, and these relations are important for the calculation of numerical tables.

Starting from the equation (F, § 2)

$$Q^\sigma_n - Q^\sigma_{n-2} = (n+\sigma-2)(n+\sigma-3)\,Q^{\sigma-2}_{n-2} - (n-\sigma+2)(n-\sigma+1)\,Q^{\sigma-2}_n,$$

we may multiply by $\cos p\theta$ or $\sin p\theta$ and integrate on both sides. Expressing the result by means of the above series, we obtain the following set of equations:

$$\left.\begin{aligned}
&(n+\sigma-1)(n-3)\,M^\sigma_n - (n-\sigma)(n+2)\,M^\sigma_{n-2} \\
&\qquad = (n+\sigma-2)(n+2)\,M^{\sigma-2}_{n-2} - (n-\sigma+1)(n-3)\,M^{\sigma-2}_n \\
&(n+\sigma)(n-4)\,S^\sigma_n - (n-\sigma-1)(n+3)\,S^\sigma_{n-2} \\
&\qquad = (n+\sigma-3)(n+3)\,S^{\sigma-2}_{n-2} - (n-\sigma+2)(n-4)\,S^{\sigma-2}_n \\
&(n+\sigma-1)(n-2)\,U^\sigma_n - (n-\sigma)(n+1)\,U^\sigma_{n-2} \\
&\qquad = (n+\sigma-2)(n+1)\,U^{\sigma-2}_{n-2} - (n-\sigma+1)(n-2)\,U^{\sigma-2}_n \\
&(n+\sigma)(n-3)\,V^\sigma_n - (n-\sigma-1)(n+2)\,V^\sigma_{n-2} \\
&\qquad = (n+\sigma-3)(n+2)\,V^{\sigma-2}_{n-2} - (n-\sigma+2)(n-3)\,V^{\sigma-2}_n
\end{aligned}\right\} \text{(K).}$$

Other relations are obtained as follows:

$$\int_{-1}^{+1} \mathrm{Q}_n^\sigma \sin p\theta \, d\mu = \int_0^\pi \mathrm{Q}_n^\sigma \sin\theta \sin p\theta \, d\theta = \frac{1}{p}\int_0^\pi \cos p\theta \frac{d}{d\theta} \mathrm{Q}_n^\sigma \sin\theta \, d\theta$$

$$= -\frac{1}{p}\int_{-1}^{+1} \cos p\theta \frac{d}{d\mu} \mathrm{Q}_n^\sigma \sin\theta \, d\mu,$$

and making use of (H_2), § 2,

$$= \frac{1}{2p\sigma}\int_{-1}^{+1} \cos p\theta \left[(\sigma+1)(n+\sigma)(n-\sigma+1)\,\mathrm{Q}_n^{\sigma-1} - (\sigma-1)\,\mathrm{Q}_n^{\sigma+1}\right] d\mu.$$

If σ be even and n odd, the integral on the left-hand side depends on S_n^σ, while that on the right-hand side depends on $\mathrm{U}_n^{\sigma-1}$ and $\mathrm{U}_n^{\sigma+1}$. For σ even and n even, the left-hand side may be expressed by M_n^σ, and the right-hand side by $\mathrm{V}_n^{\sigma-1}$ and $\mathrm{V}_n^{\sigma+1}$. Treating the integral $\int_{-1}^{+1} \mathrm{Q}_n^\sigma \cos p\theta \, d\mu$ in the same way, and collecting the results, we find:

$$\left.\begin{aligned}
2p\sigma\mathrm{M}_n^\sigma &= (\sigma+1)(n+\sigma)(n-\sigma+1)\,\mathrm{V}_n^{\sigma-1} - (\sigma-1)(n+\sigma+1)(n-\sigma)\,\mathrm{V}_n^{\sigma+1} \\
2p\sigma\mathrm{S}_n^\sigma &= (n-2)(n+3)\left[(\sigma+1)\,\mathrm{U}_n^{\sigma-1} - (\sigma-1)\,\mathrm{U}_n^{\sigma+1}\right] \\
2p\sigma\mathrm{U}_n^\sigma &= \frac{1}{(n-2)(n+3)}\left[(\sigma-1)(n+\sigma+1)(n-\sigma)\,\mathrm{S}_n^{\sigma+1}\right. \\
&\qquad\qquad \left. - (\sigma+1)(n+\sigma)(n-\sigma+1)\,\mathrm{S}_n^{\sigma-1}\right] \\
2p\sigma\mathrm{V}_n^\sigma &= (\sigma-1)\,\mathrm{M}_n^{\sigma+1} - (\sigma+1)\,\mathrm{M}_n^{\sigma-1}
\end{aligned}\right\} \text{(L).}$$

We may also connect together the different values which the same function assumes for different values of p.

Starting from the identity

$$\int_{-1}^{+1} \mathrm{Q}_n^\sigma \left[\sin(p+1)\theta - \sin(p-1)\theta\right] d\mu$$

$$= 2\int_{-1}^{+1} \mathrm{Q}_n^\sigma \sin p\theta \cos\theta \, d\mu$$

we find, transforming the right-hand side with the help of equation (A), § 2, the integral to be

$$= \frac{1}{2n+1}\int_{-1}^{+1} \left[(n-\sigma+1)\,\mathrm{Q}_{n+1}^\sigma + (n+\sigma)\,\mathrm{Q}_{n-1}^\sigma\right] \sin p\theta \, d\mu.$$

Three other equations may be obtained corresponding to this; by substituting the + for the − sign in the left-hand sign, or by changing the sine function into the cosine function. Each of these three equations gives two relations according as σ is even or odd. The following eight relations are thus obtained:

$$\frac{2n+1}{2}\left(\mathrm{S}^\sigma_n(p+1)+\mathrm{S}^\sigma_n(p-1)\right)=(n-2)\,\mathrm{M}^\sigma_{n+1}(p)+(n+3)\,\mathrm{M}^\sigma_{n-1}(p),$$

$$\frac{2n+1}{2}\left(\mathrm{M}^\sigma_n(p+1)+\mathrm{M}^\sigma_n(p-1)\right)=\frac{(n+\sigma+1)(n-\sigma+1)}{n+4}\,\mathrm{S}^\sigma_{n+1}(p)$$
$$+\frac{(n+\sigma)(n-\sigma)}{n-3}\,\mathrm{S}^\sigma_n(p),$$

$$\frac{2n+1}{2}\left(\mathrm{M}^\sigma_n(p+1)-\mathrm{M}^\sigma_n(p-1)\right)=(n+\sigma+1)(n-1)\,\mathrm{U}^{\sigma+1}_{n+1}(p)$$
$$-(n-\sigma)(n+2)\,\mathrm{U}^{\sigma+1}_{n-1}(p),$$

$$\frac{2n+1}{2}\left(\mathrm{S}^\sigma_n(p+1)-\mathrm{S}^\sigma_n(p-1)\right)=(n+\sigma+2)(n-2)\,\mathrm{V}^{\sigma+1}_{n+1}(p)$$
$$-(n-\sigma-1)(n+3)\,\mathrm{V}^{\sigma+1}_{n-1}(p),$$

$$\frac{2n+1}{2}\left(\mathrm{U}^\sigma_n(p+1)+\mathrm{U}^\sigma_n(p-1)\right)=\frac{(n+\sigma+1)(n-\sigma+1)}{n+3}\,\mathrm{V}^\sigma_{n+1}(p)$$
$$+\frac{(n+\sigma)(n-\sigma)}{n-2}\,\mathrm{V}^\sigma_{n-1}(p),$$

$$\frac{2n+1}{2}\left(\mathrm{V}^\sigma_n(p+1)+\mathrm{V}^\sigma_n(p-1)\right)=(n-1)\,\mathrm{U}^\sigma_{n+1}(p)+(n+2)\,\mathrm{U}^\sigma_{n-1}(p),$$

$$\frac{2n+1}{2}\left(\mathrm{U}^\sigma_n(p-1)-\mathrm{U}^\sigma_n(p+1)\right)=\frac{n+\sigma+1}{n+3}\,\mathrm{M}^{\sigma+1}_{n+1}(p)-\frac{n-\sigma}{n-2}\,\mathrm{M}^{\sigma+1}_{n-1}(p),$$

$$\frac{2n+1}{2}\left(\mathrm{V}^\sigma_n(p-1)-\mathrm{V}^\sigma_n(p+1)\right)=\frac{n+\sigma+2}{n+4}\,\mathrm{S}^{\sigma+1}_{n+1}(p)-\frac{n-\sigma-1}{n-3}\,\mathrm{S}^{\sigma+1}_{n-1}(p).$$

§ 10. *Numerical Calculations of the Four Series.*

For the calculation of the numerical values of the series $\mathrm{M}^\sigma_n(p)$, the special case $\sigma = 0$ was taken as starting-point. In this particular case the well-known integral

$$\int_{-1}^{+1}\mathrm{P}_n\sin p\theta\,d\mu=-\pi p\,\frac{(n+p-2)!!\,(n-p-2)!!}{(n+p+1)!!\,(n-p+1)!!},$$

which holds when $n+p$ is odd and $n \gtreqless p-1$, gives

$$\mathrm{M}^0_n=-\frac{n!!\,(n+2)!!}{(n-1)!!\,(n-3)!!}\,p\,\frac{(n+p-2)!!\,(n-p-2)!!}{(n+p+1)!!\,(n-p+1)!!}\ \text{when } n \text{ is even and } p \text{ odd.}$$

The values of M^0_n were calculated by this equation for all values of p up to and including $p = 11$, and for all values of n up to and including 12. The first of the equations K, § 9, was then applied, and substituting $\sigma = 2$, $n = 2$, and $\mathrm{M}^2_0 = 0$, the value of M^2_2 was calculated. Similarly putting $n = 4, 6$, &c., the values of M^2_n for even values of n were all tabulated. By successive steps I similarly found M^4_n, M^6_n, &c., up to M_{12}^{12}. The last value so obtained, and some of the intermediate ones, were calculated independently from the series defining M, numerical errors being easily detected in this way. The values of S_n^σ, U_n^σ, V_n^σ were dealt with in a similar manner. As a final check, the values of S_n^σ and V_n^σ were obtained from U_n^σ and M_n^σ by means of the second and fourth of the equations (L).

§ 11. *Application of the Previous Results to the Expression of Functions by Means of a Series of Spherical Harmonics.*

The results obtained in this investigation lead to a new method of calculating the coefficients of the series of spherical harmonics which represent a function F of spherical co-ordinates. If the values of this function are given for all points of a sphere, the coefficients of the term involving T_n^σ is known to depend solely on the integral $\int FT_n^\sigma \, dS$ taken over the surface of the sphere. This theoretically perfect proceeding has to be modified in practice when the values of F are known only at definite points, so that the intermediate positions have to be evaluated by a process of interpolation on the supposition that F is continuous everywhere.

All methods which have been applied so far, suffer from the serious inconvenience that the method of least squares is applied in such a way as to make the value of the coefficients of lower degrees depend on the number of terms which are taken into account. That is to say, the coefficients are not independent of each other as they ought to be.

A method suggested by F. E. NEUMANN, which is free from this defect, introduces other complications, and has never, as far as I know, been applied in practice.

The theorem of § 8 offers a simple solution of the practical difficulties, and reduces the whole problem to an expansion by means of FOURIER's analysis, which can be carried out either by the well-known process of calculation, or by mechanical means.

Let F be expressed, in the first place, for different circles of latitude as a series proceeding by cosines and sines of the longitude and of its multiples. This first step is common to all methods.

The result may be expressed symbolically by

$$F = \kappa^0 + \kappa' \cos \phi + \kappa^2 \cos 2\phi + \ldots \qquad + K' \sin \phi + K^2 \sin 2\phi + \ldots ,$$

where κ^0, κ', K', $K^2 \ldots$ are functions of the colatitude. If their values are known at $q - 1$ equidistant circles of latitude, and at the poles, we may determine the coefficients a_p^σ and α_p^σ, which satisfy the equations

$$\kappa^\sigma = a_0^\sigma + a_1^\sigma \cos \theta + a_2^\sigma \cos 2\theta + \ldots a_p^\sigma \cos p\theta + \ldots a_q^\sigma \cos q\theta,$$

$$K^\sigma = \alpha_0^\sigma + \alpha_1^\sigma \cos \theta + \alpha_2^\sigma \cos 2\theta + \ldots \alpha_p^\sigma \cos p\theta + \ldots \alpha_q^\sigma \cos q\theta.$$

If we give to θ the successive values 0, π/q, $2\pi/q \ldots q\pi/q$, we shall have $q + 1$

equations to determine $q+1$ coefficients. The solution is conducted according to well-known rules, making use of the proposition that, if p and s are integer numbers smaller than q,

$$0 = \tfrac{1}{2} + \cos\frac{p\pi}{q}\cos\frac{s\pi}{q} + \cos\frac{2p\pi}{q}\cos\frac{2s\pi}{q} + \ldots$$

$$+ \cos\frac{(q-1)p\pi}{q}\cos\frac{(q-1)s\pi}{q} + \tfrac{1}{2}\cos p\pi\cos s\pi.$$

If p and s are equal to each other the sum on the right-hand side is equal to $\tfrac{1}{2}q$.

The factor $\tfrac{1}{2}$ of the first and last term should be noted. If we designate by $\kappa_0, \kappa_1, \kappa_2 \ldots \kappa_q$, the value of κ for $\theta = 0$, and at the successive circles of latitude, it follows that

$$\frac{q}{2}\,a_p^\sigma = \tfrac{1}{2}(\kappa_0^\sigma + \kappa_q^\sigma\cos p\pi) + \sum_{s=1}^{s=q-1}\kappa_s^\sigma\cos sp\pi/q.$$

The condition under which the coefficients are determined is that the coefficients of the Fourier series higher than a_q^s are zero. It will be noted that, even if we suppose some of the lower coefficients to vanish, the equations will still give *those* values for the remaining coefficients which, according to the method of least squares, fit in best with the assumed values of κ, but that in the calculations half weight only is given to the values at the two poles.

[The above method of obtaining the coefficients a_p^σ, which is identical with that in common use, when the range of θ is 2π, is convenient whenever the function to be analysed is known at every point of the sphere, so that the coefficients κ and K may be determined for a sufficient number of equidistant circles of latitude. But other methods are available, and hence the process of obtaining the coefficients of the series of spherical harmonics which I am endeavouring to explain, is not restricted to cases where the original function is known everywhere; provided it is continuous, as well as its derivatives over the surface of the sphere. Graphical interpolation may be employed to determine the function at unknown points with sufficient accuracy, and there are several good mechanical devices in existence, by means of which at any rate the lowest and most important coefficients of the Fourier series may be found. It will, in some cases, materially help this process of interpolation if it is remembered that continuity at the pole involves the vanishing of all the values of κ and K except κ^0.—*August* 2, 1902.]

Having calculated the coefficients, the reduction to spherical harmonics is made, by substitution of

$$\cos p\theta = A_\sigma^\sigma Q_\sigma^\sigma + A_{\sigma+1}^\sigma Q_{\sigma+1}^\sigma + \ldots \qquad A_{p-1}^\sigma Q_{p-1}^\sigma + \ldots \qquad A_n^\sigma Q_n^\sigma + \ldots,$$

where the values of A_n^σ may be calculated and tabulated once for all. By the theorem of § 8, all coefficients vanish up to and inclusive of A_{p-2}^σ if σ be odd, so that the spherical harmonic of degree n will only depend on the Fourier coefficients a_p, for which p is equal to or smaller than $n+1$. When σ is even we secure the same advantage by developing κ^σ and K^σ in terms of the sine functions, and we write in that case:

$$\kappa^\sigma = b_1^\sigma \sin\theta + b_2^\sigma \sin 2\theta + \ldots \qquad b_p^\sigma \sin p\theta + \ldots$$

$$K^\sigma = \beta_1^\sigma \sin\theta + \beta_2^\sigma \sin 2\theta + \ldots \qquad \beta_p^\sigma \sin p\theta + \ldots$$

and

$$\sin p\theta = B_{p-1}^\sigma Q_{n-1}^\sigma + B_{p+1}^\sigma Q_{n+1}^\sigma + \ldots \qquad B_n^\sigma Q_n^\sigma + \ldots$$

To calculate finally a coefficient such as B_n^σ we proceed in the usual way, thus:

$$\int_{-1}^{+1} Q_n^\sigma \sin p\theta \, d\mu = \frac{2B_n^\sigma}{2n+1} \frac{(n+\sigma)!}{(n-\sigma)!}.$$

Therefore

$$B_n^\sigma = \frac{2n+1}{2} \frac{(n-\sigma-1)!!\,(n-3)!!}{(n+\sigma)!!\,(n+2)!!} \pi M_n^\sigma \text{ when } n \text{ is even and } \sigma \text{ is even}$$

$$= \frac{2n+1}{2} \frac{(n-\sigma)!!\,(n-4)!!}{(n+\sigma-1)!!\,(n+3)!!} \pi S_n^\sigma \text{ when } n \text{ is odd and } \sigma \text{ is even.}$$

Similarly

$$A_n^\sigma = \frac{2n+1}{2} \frac{(n-\sigma)!!\,(n-3)!!}{(n+\sigma-1)!!\,(n+2)!!} \pi V_n^\sigma \text{ when } n \text{ is even and } \sigma \text{ is odd}$$

$$= \frac{2n+1}{2} \frac{(n-\sigma-1)!!\,(n-2)!!}{(n+\sigma)!!\,(n+1)!!} \pi U_n^\sigma \text{ when } n \text{ is odd and } \sigma \text{ is odd.}$$

Combining equations, we find that if g_n^σ denote the coefficient of $t_n^\sigma Q_n^\sigma \cos\sigma\phi$ in the development of F, and h_n^σ the coefficient of $t_n^\sigma Q_n^\sigma \sin\sigma\phi$,

$$g_n^\sigma = \frac{2n+1}{2t_n^\sigma} \frac{(n-\sigma)!!\,(n-3)!!}{(n+\sigma-1)!!(n+2)!!} \pi \sum_{p=1}^{p=n+1}{}^{(2)} a_p^\sigma V_n^\sigma(p), \text{ when } n \text{ is even and } \sigma \text{ odd and } p \text{ odd}$$

$$= \frac{2n+1}{2t_n^\sigma} \frac{(n-\sigma-1)!!\,(n-2)!!}{(n+\sigma)!!\,(n+1)!!} \pi \sum_{p=0}^{p=n+1}{}^{(2)} a_p^\sigma U_p^\sigma(p), \text{ when } n \text{ is odd and } \sigma \text{ odd and } p \text{ even}$$

$$= \frac{2n+1}{2t_n^\sigma} \frac{(n-\sigma-1)!!\,(n-3)!!}{(n+\sigma)!!\,(n+2)!!} \pi \sum_{p=1}^{p=n+1}{}^{(2)} b_p^\sigma M_n^\sigma(p) \text{ when } n \text{ is even and } \sigma \text{ even and } p \text{ odd}$$

$$= \frac{2n+1}{2t_n^\sigma} \frac{(n-\sigma)!!\,(n-4)!!}{(n+\sigma-1)!!\,(n+3)!!} \pi \sum_{p=2}^{p=n+1}{}^{(2)} b_p^\sigma S_n^\sigma(p) \text{ when } n \text{ is odd and } \sigma \text{ even and } p \text{ even.}$$

To obtain h_n^σ, substitute α and β for a and b.

The coefficient t_n^σ is introduced because the quantity here designated by Q_n^σ is not uniformly accepted as the standard form for a spherical harmonic. For some purposes it is more convenient to take the form of Gauss, for which Thomson and Tait have adopted the symbol Θ_n^σ, and which is connected with Q_n^σ by the relation

$$\Theta_n^\sigma = \frac{(n-\sigma)!}{(2n-1)!!} Q_n^\sigma.$$

Another constant, which has been applied by Adolf Schmidt, is based on the consideration that for numerical work it is inconvenient to deal with functions the average values of which differ considerably from each other. This author therefore takes a function $r_n^\sigma \Theta_n^\sigma$ as basis of calculation, and determines r_n^σ, so that the average value of the square of $r_n^\sigma \Theta_n^\sigma \cos \sigma\phi$ over a sphere of unit radius is equal to unity. The function R_n^σ defined in this way is connected with Θ_n^σ and Q_n^σ by

$$R_n^\sigma = (2n-1)!! \sqrt{\frac{\epsilon_\sigma (2n+1)}{(n+\sigma)!\,(n-\sigma)!}}\, \Theta_n^\sigma = \sqrt{\frac{\epsilon_\sigma (2n+1)(n-\sigma)!}{(n+\sigma)!}}\, Q_n^\sigma,$$

where ϵ_σ is equal to 1 for $\sigma = 0$ and equal to 2 for all other values of σ.

If a function is to be expanded in terms of R_n^σ, we must therefore write

$$t_n^\sigma = \sqrt{\frac{\epsilon_\sigma (2n+1)(n-\sigma)!}{(n+\sigma)!}}.$$

For numerical work the introduction of R_n^σ in place of Q_n^σ possesses undoubted advantages. Although at first I was reluctant to adopt the additional complication due to the introduction of a square root and the addition of yet another function to those given by previous writers, I found that the inconvenience of tabulating values differing considerably in magnitude from each other was very great, and I therefore felt myself almost compelled during the course of the investigation to adopt Schmidt's function as above defined.

We now substitute t_n^σ into the equations for g_n^σ and h_n^σ and write:

$$v_n^\sigma = \frac{\pi}{2}\frac{(n-3)!!}{(n+2)!!}\sqrt{\frac{2n+1}{\epsilon_\sigma}}\sqrt{\frac{(n+\sigma)!!\,(n-\sigma)!!}{(n+\sigma-1)!!\,(n-\sigma-1)!!}}\, V_n^\sigma,$$

$$u_n^\sigma = \frac{\pi}{2}\frac{(n-2)!!}{(n+1)!!}\sqrt{\frac{2n+1}{\epsilon_\sigma}}\sqrt{\frac{(n+\sigma-1)!!\,(n-\sigma-1)!!}{(n+\sigma)!!\,(n-\sigma)!!}}\, U_n^\sigma,$$

$$m_n^\sigma = \frac{\pi}{2}\frac{(n-3)!!}{(n+2)!!}\sqrt{\frac{2n+1}{\epsilon_\sigma}}\sqrt{\frac{(n+\sigma-1)!!\,(n-\sigma-1)!!}{(n+\sigma)!!\,(n-\sigma)!!}}\, M_n^\sigma,$$

$$s_n^\sigma = \frac{\pi}{2}\frac{(n-4)!!}{(n+3)!!}\sqrt{\frac{2n+1}{\epsilon_\sigma}}\sqrt{\frac{(n+\sigma)!!\,(n-\sigma)!!}{(n+\sigma-1)!!\,(n-\sigma-1)!!}}\, S_n^\sigma,$$

where $\epsilon_0 = 1$ and $\epsilon_\sigma = 2$ if $\sigma > 0$.

By substitution of the values of V_n^σ, &c., we may also write more simply

$$v_n^\sigma = \frac{1}{2\epsilon_\sigma}\int_{-1+1}^{+1} \mathrm{R}_n^\sigma \cos p\theta \, d\mu \,; \text{ if } \sigma \text{ be odd, } p \text{ odd, } n \text{ even.}$$

$$u_n^\sigma = \frac{1}{2\epsilon_\sigma}\int_{-1+1} \mathrm{R}_n^\sigma \cos p\theta \, d\mu \,; \text{ if } \sigma \text{ ,, } \quad \text{ ,, } \quad p \text{ even, } n \text{ odd.}$$

$$m_n^\sigma = \frac{1}{2\epsilon_\sigma}\int_{-1} \mathrm{R}_n^\sigma \sin p\theta \, d\mu \,; \text{ if } \sigma \text{ be even, } p \text{ odd, } n \text{ even.}$$

$$s_n^\sigma = \frac{1}{2\epsilon_0}\int_{-1}^{+1} \mathrm{R}_n^\sigma \sin p\theta \, d\mu \,; \text{ if } \sigma \text{ ,, } \quad \text{ ,, } \quad p \text{ even, } n \text{ odd.}$$

It would thus appear to be unnecessary to have separate symbols for u_n^σ and v_n^σ or for m_n^σ and s_n^σ, but as it is convenient to tabulate separately the integrals for odd and even values of p, the retention of all four symbols facilitates reference.

We may now obtain the final coefficients by summation thus:

$$g_n^\sigma = \sum_{p=1}^{p=n+1}{}^{(2)} \, a_p^\sigma v_n^\sigma, \text{ when } n \text{ is even, } \sigma \text{ odd, and } p \text{ odd,}$$

$$= \sum_{p=0}^{p=n+1}{}^{(2)} \, a_p^\sigma u_n^\sigma, \quad \text{,,} \quad n \text{ ,, odd, } \sigma \text{ ,, } \quad \text{,, } p \text{ even,}$$

$$= \sum_{p=1}^{p=n+1}{}^{(2)} \, b_p^\sigma m_n^\sigma, \quad \text{,,} \quad n \text{ ,, even, } \sigma \text{ even, ,, } p \text{ odd,}$$

$$= \sum_{p=2}^{p=n+1}{}^{(2)} \, b_p^\sigma s_n^\sigma, \quad \text{,,} \quad n \text{ ,, odd, } \sigma \text{ ,, } \quad \text{,, } p \text{ even,}$$

with similar equations for h_n^σ, a and b being replaced by α and β.

The values of v_n^σ, u_n^σ, m_n^σ and s_n^σ are given in Tables V.—VIII. and their logarithms in Tables IX.—XII., for values of n, σ and p up to 12 inclusive. By means of these tables we may, for instance, write down at once the coefficients as far as the third degree as follows:

$$g_0^{\circ} = \cdot 785398 b_1^{\circ}$$
$$g_1^{\circ} = \cdot 680175 b_2^{\circ}$$
$$g_2^{\circ} = - \cdot 219525 b_1^{\circ} + \cdot 658575 b_3^{\circ}$$
$$g_3^{\circ} = - \cdot 259746 b_2^{\circ} + \cdot 649365 b_4^{\circ}$$
$$g_1' = \cdot 680175 a_0' - \cdot 340087 a_2'$$
$$g_2' = \cdot 380229 a_1' - \cdot 380229 a_3'$$
$$g_3' = \cdot 159061 a_0' + \cdot 318123 a_2' - \cdot 397653 a_4'$$
$$g_2^2 = \cdot 570345 b_1^2 - \cdot 190115 b_3^2$$
$$g_3^2 = \cdot 502996 b_2^2 - \cdot 251498 b_4^2$$
$$g_3^3 = \cdot 616042 a_0^3 - \cdot 410695 a_2^3 + \cdot 102672 a_4^3.$$

The calculation of the ten coefficients g up to the third degree involves therefore the determination of sixteen Fourier coefficients. There is no h coefficient of zero type, and there are therefore only six h coefficients up to the third degree which depend upon twelve Fourier coefficients. In general there will be $\frac{1}{2}(n+1)(n+2)$ g coefficients, and $\frac{1}{2}(n+1)n$, h coefficients up to and including the n^{th} degree, giving a total of $(n+1)^2$ coefficients in the series of spherical harmonics. To determine these completely, it would be necessary to calculate $(n+1)^2$ Fourier coefficients for the quantities g and $n\ n+1$ for the quantities h, giving a total of $(n+1)(n+2)$ Fourier coefficients. The tables accompanying this paper being calculated for values of n up to 12 are therefore sufficient to determine 169 coefficients in the series of spherical harmonics, by means of a simple computation after the determination of 182 coefficients of the Fourier series. These latter coefficients may be evaluated by mechanical devices.

§ 12. *Special Application to the Theory of Terrestrial Magnetism.*

The advantages of the proposed method of obtaining the coefficients of a series of spherical harmonics are considerably increased when the quantities to be represented by such a series are not given directly, but by means of their differential coefficients. This is the case when the magnetic potential has to be calculated by means of the observed magnetic forces. If X and Y represent the components of magnetic force resolved towards the geographical north and east respectively, the magnetic potential is determined by means of the equations

$$dV/d\theta = X; \quad dV/d\phi = -Y \sin\theta.$$

If no electric currents of sufficient intensity traverse the earth's surface, a function V can be found which satisfies both equations. If Y $\sin\theta$ be obtained in a series proceeding by spherical harmonics, then all the terms which depend on the longitude ϕ are at once expressed in a similar series, as the integration according to ϕ leaves each term in the standard form. The treatment of the other component acting along the meridian involves, however, serious difficulties, and it is not necessary here to enter into the question as to the more or less complicated methods by means of which V has been hitherto derived in the standard form from X.

The method based on the results of the preceding investigation avoids these difficulties. For odd values of σ, X is expressed in a series, each term of which has the form $\cos n\theta \sin m\phi$ or $\cos n\theta \cos m\phi$, while for even values of σ, X is expressed in terms of the form $\sin n\theta \sin m\phi$ and $\sin n\theta \cos m\phi$. After integration with respect to θ, the formula of § 11 will determine the required coefficients. We may treat the eastern force similarly, obtaining a series proceeding by $\cos p\theta$ or $\sin p\theta$ according as σ is odd or even. Y $\sin\theta$ is then derived in a series proceeding by sin $p\theta$ or cos $p\theta$, and after integration with respect to ϕ the equations of § 11 may be applied. The spheroidal shape of the earth may also, if necessary, be easily taken into account, but I reserve the discussion of this matter until the completion of some calculations, on which I am at present engaged, will allow me to compare the results obtained by the method I advocate with those obtained in other ways.

Great pains have been taken to guard against numerical errors, and it is hoped that all numbers given in the tables are correct, with the proviso, however, that the last decimal place in Tables V. and XII. is uncertain by one unit, and wrong possibly even occasionally by two units. Professor Core has assisted me in the numerical work.

TABLE I.—Values of $V_n^\sigma(p)$.

	n; p:	1.	3.	5.	7.	9.	11.
$\sigma=1$	2	1	-1	—	—	—	—
	4	1	$\frac{5}{2}$	$-\frac{7}{2}$	—	—	—
	6	1	$\frac{7}{5}$	$\frac{21}{5}$	$-\frac{33}{5}$	—	—
	8	1	$\frac{6}{5}$	$\frac{66}{35}$	$\frac{429}{70}$	$-\frac{143}{14}$	—
	10	1	$\frac{55}{49}$	$\frac{143}{98}$	$\frac{715}{294}$	$\frac{2431}{294}$	$-\frac{4199}{294}$
	12	1	$\frac{13}{12}$	$\frac{325}{252}$	$\frac{221}{126}$	$\frac{4199}{1386}$	$\frac{4199}{396}$
$\sigma=3$	4	3	$-\frac{9}{2}$	$\frac{3}{2}$	—	—	—
	6	3	1	$-\frac{23}{3}$	$\frac{11}{3}$	—	—
	8	3	2	$-\frac{2}{3}$	$-\frac{65}{6}$	$\frac{13}{2}$	—
	10	3	$\frac{117}{49}$	$\frac{93}{98}$	$-\frac{213}{98}$	$-\frac{1377}{98}$	$\frac{969}{98}$
	12	3	$\frac{31}{12}$	$\frac{179}{108}$	$-\frac{17}{270}$	$-\frac{323}{90}$	$-\frac{9367}{540}$
$\sigma=5$	6	5	-9	5	-1	—	—
	8	5	-2	-10	$\frac{19}{2}$	$-\frac{5}{2}$	—
	10	5	$\frac{5}{7}$	$-\frac{83}{14}$	$-\frac{19}{2}$	$\frac{1003}{70}$	$-\frac{323}{70}$
	12	5	$\frac{25}{12}$	$-\frac{35}{12}$	$-\frac{239}{30}$	$-\frac{247}{30}$	$\frac{1159}{60}$
$\sigma=7$	8	7	-14	10	$-\frac{7}{2}$	$\frac{1}{2}$	—
	10	7	$-\frac{41}{7}$	$-\frac{145}{14}$	$\frac{31}{2}$	$-\frac{107}{14}$	$\frac{19}{14}$
	12	7	$-\frac{7}{4}$	$-\frac{121}{12}$	$-\frac{133}{30}$	$\frac{193}{10}$	$-\frac{763}{60}$
$\sigma=9$	10	9	$-\frac{135}{7}$	$\frac{225}{14}$	$-\frac{15}{2}$	$\frac{27}{14}$	$-\frac{3}{14}$
	12	9	$-\frac{41}{4}$	$-\frac{325}{36}$	$\frac{371}{18}$	$-\frac{29}{2}$	$\frac{173}{36}$
$\sigma=11$	12	11	$-\frac{99}{4}$	$\frac{275}{12}$	$-\frac{77}{6}$	$\frac{9}{2}$	$-\frac{11}{12}$

Table II.—Values of $U_n^\sigma(p)$.

	n; p:	0.	2.	4.	6.	8.	10.	12.
$\sigma=1$	1	1	$-\frac{1}{2}$	—	—	—	—	—
	3	1	2	$-\frac{5}{2}$	—	—	—	—
	5	1	$\frac{5}{4}$	$\frac{7}{2}$	$-\frac{21}{4}$	—	—	—
	7	1	$\frac{28}{25}$	$\frac{42}{25}$	$\frac{132}{25}$	$-\frac{429}{50}$	—	—
	9	1	$\frac{15}{14}$	$\frac{66}{49}$	$\frac{429}{196}$	$\frac{715}{98}$	$-\frac{2431}{196}$	—
	11	1	$\frac{22}{21}$	$\frac{715}{588}$	$\frac{715}{441}$	$\frac{2431}{882}$	$\frac{4199}{441}$	$-\frac{4199}{252}$
$\sigma=3$	3	3	-2	$\frac{1}{2}$	—	—	—	—
	5	3	$\frac{7}{4}$	$-\frac{11}{2}$	$\frac{9}{4}$	—	—	—
	7	3	$\frac{12}{5}$	$\frac{2}{15}$	$-\frac{44}{5}$	$\frac{143}{30}$	—	—
	9	3	$\frac{37}{14}$	$\frac{10}{7}$	$-\frac{39}{28}$	$-\frac{169}{14}$	$\frac{221}{28}$	—
	11	3	$\frac{58}{21}$	$\frac{167}{84}$	$\frac{3}{7}$	$-\frac{17}{6}$	$-\frac{323}{21}$	$\frac{323}{28}$
$\sigma=5$	5	5	$-\frac{15}{4}$	$\frac{3}{2}$	$-\frac{1}{4}$	—	—	—
	7	5	$\frac{4}{5}$	$-\frac{38}{5}$	$\frac{28}{5}$	$-\frac{13}{10}$	—	—
	9	5	$\frac{5}{2}$	$-\frac{26}{7}$	$-\frac{241}{28}$	$\frac{145}{14}$	$-\frac{85}{28}$	—
	11	5	$\frac{10}{3}$	$-\frac{95}{84}$	$-\frac{137}{21}$	$-\frac{1717}{210}$	$\frac{323}{21}$	$-\frac{323}{60}$
$\sigma=7$	7	7	$-\frac{28}{5}$	$\frac{14}{5}$	$-\frac{4}{5}$	$\frac{1}{10}$	—	—
	9	7	$-\frac{1}{2}$	$-\frac{62}{7}$	$\frac{253}{28}$	$-\frac{53}{14}$	$\frac{17}{28}$	—
	11	7	2	$-\frac{199}{28}$	$-\frac{127}{21}$	$\frac{199}{14}$	$-\frac{57}{7}$	$\frac{19}{12}$
$\sigma=9$	9	9	$-\frac{15}{2}$	$\frac{30}{7}$	$-\frac{45}{28}$	$\frac{5}{14}$	$-\frac{1}{28}$	—
	11	9	-2	$-\frac{265}{28}$	$\frac{85}{7}$	$-\frac{293}{42}$	$\frac{43}{21}$	$-\frac{1}{4}$
$\sigma=11$	11	11	$-\frac{66}{7}$	$\frac{165}{28}$	$-\frac{55}{21}$	$\frac{11}{14}$	$-\frac{1}{7}$	$\frac{1}{84}$

TABLE III.—Values of $M_n^\sigma(p)$.

	n ; p :	1.	3.	5.	7.	9.	11.
$\sigma=0$.	0	-1	—	—	—	—	—
	2	-1	3	—	—	—	—
	4	-1	$-\frac{15}{2}$	$\frac{35}{2}$	—	—	—
	6	-1	$-\frac{21}{5}$	-21	$\frac{231}{5}$	—	—
	8	-1	$-\frac{18}{5}$	$-\frac{66}{7}$	$-\frac{429}{10}$	$\frac{1287}{14}$	—
	10	-1	$-\frac{165}{49}$	$-\frac{715}{98}$	$-\frac{715}{42}$	$-\frac{7293}{98}$	$\frac{46189}{294}$
	12	-1	$-\frac{13}{4}$	$-\frac{1625}{252}$	$-\frac{221}{18}$	$-\frac{4199}{154}$	$-\frac{4199}{36}$
$\sigma=2$.	2	3	-1	—	—	—	—
	4	3	$\frac{33}{2}$	$-\frac{21}{2}$	—	—	—
	6	3	11	39	-33	—	—
	8	3	10	22	$\frac{143}{2}$	$-\frac{143}{2}$	—
	10	3	$\frac{471}{49}$	$\frac{1833}{98}$	$\frac{507}{14}$	$\frac{11271}{98}$	$-\frac{12597}{98}$
	12	3	$\frac{113}{12}$	$\frac{625}{36}$	$\frac{527}{18}$	$\frac{323}{6}$	$\frac{6137}{36}$
$\sigma=4$.	4	15	$-\frac{15}{2}$	$\frac{3}{2}$	—	—	—
	6	15	31	-37	11	—	—
	8	15	38	34	$-\frac{169}{2}$	$\frac{65}{2}$	—
	10	15	$\frac{285}{7}$	$\frac{723}{14}$	$\frac{375}{14}$	$-\frac{2091}{14}$	$\frac{969}{14}$
	12	15	$\frac{505}{12}$	$\frac{715}{12}$	$\frac{1717}{30}$	$\frac{323}{30}$	$-\frac{13889}{60}$
$\sigma=6$.	6	35	-21	7	-1	—	—
	8	35	42	-74	$\frac{79}{2}$	$-\frac{15}{2}$	—
	10	35	$\frac{465}{7}$	$\frac{17}{14}$	$-\frac{1765}{14}$	$\frac{1377}{14}$	$-\frac{323}{14}$
	12	35	$\frac{315}{4}$	$\frac{635}{12}$	$-\frac{1607}{30}$	$-\frac{1691}{10}$	$\frac{11039}{60}$
$\sigma=8$.	8	63	-42	18	$-\frac{9}{2}$	$\frac{1}{2}$	—
	10	63	$\frac{333}{7}$	$-\frac{1629}{14}$	$\frac{1191}{14}$	$-\frac{411}{14}$	$\frac{57}{14}$
	12	63	$\frac{371}{4}$	$-\frac{565}{12}$	$-\frac{863}{6}$	$\frac{1079}{6}$	$-\frac{973}{12}$
$\sigma=10$.	10	99	$-\frac{495}{7}$	$\frac{495}{14}$	$-\frac{165}{14}$	$\frac{33}{14}$	$-\frac{3}{14}$
	12	99	$\frac{187}{4}$	$-\frac{1925}{12}$	$\frac{869}{6}$	$-\frac{413}{6}$	$\frac{211}{12}$
$\sigma=12$.	12	143	$-\frac{429}{4}$	$\frac{715}{12}$	$-\frac{143}{6}$	$\frac{13}{2}$	$-\frac{13}{12}$

TABLE IV.—Values of $S_n^\sigma(p)$.

	n; p:	2	4.	6.	8.	10.	12.
$\sigma=0$	1	-2	—	—	—	—	—
	3	-2	5	—	—	—	—
	5	-2	$-\frac{56}{5}$	$\frac{126}{5}$	—	—	—
	7	-2	-6	$-\frac{198}{7}$	$\frac{1287}{21}$	—	—
	9	-2	$-\frac{176}{35}$	$-\frac{429}{35}$	$-\frac{1144}{21}$	$\frac{2431}{21}$	—
	11	-2	$-\frac{65}{14}$	$-\frac{65}{7}$	$-\frac{442}{21}$	$-\frac{20995}{231}$	$\frac{4199}{22}$
$\sigma=2$	3	6	-3	—	—	—	—
	5	6	24	-18	—	—	—
	7	6	$\frac{46}{3}$	$\frac{154}{3}$	$-\frac{143}{3}$	—	—
	9	6	$\frac{96}{7}$	$\frac{195}{7}$	$\frac{624}{7}$	$-\frac{663}{7}$	—
	11	6	$\frac{183}{14}$	$\frac{163}{7}$	$\frac{306}{7}$	$\frac{969}{7}$	$-\frac{323}{2}$
$\sigma=4$	5	30	-24	6	—	—	—
	7	30	$\frac{110}{3}$	$-\frac{190}{3}$	$\frac{65}{3}$	—	—
	9	30	48	33	-120	51	—
	11	30	$\frac{105}{2}$	57	$\frac{102}{5}$	$-\frac{969}{5}$	$\frac{969}{10}$
$\sigma=6$	7	70	-70	30	-5	—	—
	9	70	32	-123	80	-17	—
	11	70	$\frac{145}{2}$	-27	$-\frac{842}{5}$	$\frac{779}{5}$	$-\frac{399}{10}$
$\sigma=8$	9	126	-144	81	-24	3	—
	11	126	$\frac{9}{2}$	-183	174	-69	$\frac{21}{2}$
$\sigma=10$	11	198	$-\frac{495}{2}$	165	-66	15	$-\frac{3}{2}$

TABLE V.—$v_n^\sigma = \frac{1}{4}\int_{-1}^{+1} \mathrm{R}_n^\sigma \cos p\theta \, d\mu$.

	n ; p :	1.	3.	5.	7.	9.	11.
$\sigma = 1$	2	·380229	– ·380229	—	—	—	—
	4	·116421	·291052	– ·407473	—	—	—
	6	·063364	·088709	·266128	– ·418201	—	—
	8	·041506	·049808	·078269	·254374	– ·423957	—
	10	·029936	·033601	·043681	·072803	·247529	– ·427550
	12	·022921	·024831	·029561	·040203	·069442	·243047
$\sigma = 3$	4	·308021	– ·462031	·154011	—	—	—
	6	·180337	·060112	– ·460860	·220412	—	—
	8	·120909	·080606	– ·026869	– ·436615	·261969	—
	10	·088128	·070143	·027877	– ·063848	– ·412763	·290463
	12	·067865	·058439	·037493	– ·001424	– ·081187	– ·392403
$\sigma = 5$	6	·257385	– ·463293	·257385	– ·051477	—	—
	8	·187600	– ·075040	– ·375200	·356440	– 093800	—
	10	·140757	·020108	– ·166898	– ·267439	·403370	– ·129899
	12	·109922	·045801	– ·064121	– ·175142	– ·181005	·424665
$\sigma = 7$	8	·221971	– ·443943	·317102	– ·110986	·015855	—
	10	·181681	– ·152018	– ·268813	·402293	– ·198366	·035224
	12	·146379	– ·036595	– ·210856	– ·092707	·403588	– ·265922
$\sigma = 9$	10	·195951	– ·419896	·349913	– ·163293	·041990	– ·004665
	12	·172490	– ·196447	– ·173022	·395023	– ·277900	·092101
$\sigma = 11$	12	·176003	– ·396007	·366673	– ·205337	·072001	– ·014667

Table VI.—$u_n^\sigma = \frac{1}{4}\int_{-1}^{+1} \mathrm{R}_n^\sigma \cos p\theta \, d\mu$.

	n; p:	0.	2.	4.	6.	8.	10.	12.
$\sigma = 1$	1	·680175	− ·340087	—	—	—	—	—
	3	·159061	·318123	− ·397653	—	—	—	—
	5	·078817	·098522	·275861	− ·413791	—	—	—
	7	·049121	·055015	·082523	·259357	− ·421455	—	—
	9	·034342	·036794	·046256	·075166	·250553	− ·425940	—
	11	·025739	·026965	·031298	·041731	·070943	·245076	− ·428884
$\sigma = 3$	3	·616042	− ·410695	·102672	—	—	—	—
	5	·255398	·148982	− ·468228	·191548	—	—	—
	7	·153143	·122515	·006806	− ·449220	·243328	—	—
	9	·105449	·092895	·050214	− ·048959	− ·424306	·277431	—
	11	·078434	·072209	·051978	·011205	− ·074077	− ·402128	·301596
$\sigma = 5$	5	·571086	− ·428315	·171326	− ·028554	—	—	—
	7	·282177	·045148	− ·428908	·316038	− ·073366	—	—
	9	·185519	·092760	− ·137814	− ·319358	·384290	− ·112637	—
	11	·135311	·090207	− ·030606	− ·176548	− ·221265	·416242	− ·145685
$\sigma = 7$	7	·538359	− ·430688	·215344	− ·061527	·007691	—	—
	9	·290384	− ·020742	− ·367424	·374832	− ·157044	·025186	—
	11	·201669	·057620	− ·204756	− ·174231	·409512	− ·234595	·045616
$\sigma = 9$	9	·513121	− ·427601	·244343	− ·091629	·020362	− ·002036	—
	11	·291820	− ·064849	− ·306874	·393725	− ·226199	·066393	− ·008106
$\sigma = 11$	11	·492809	− ·422408	·264005	− ·117335	·035201	− ·006400	·000533

TABLE VII.—$m_n^\sigma = \frac{1}{2\epsilon_\sigma}\int_{-1}^{+1} R_n^\sigma \sin p\theta \, d\mu$; $\epsilon_0 = 1$; $\epsilon_\sigma = 2$ if $\sigma > 0$.

	n ; p :	1.	3.	5.	7.	9.	11.
$\sigma = 0$	0	·785398	—	—	—	—	—
	2	−·219525	·658575	—	—	—	—
	4	−·036816	−·276117	·644271	—	—	—
	6	−·013827	−·058074	−·290369	·638812	—	—
	8	−·006918	−·024904	−·065224	−·296770	·635936	—
	10	−·004037	−·013592	−·029450	−·068717	−·300391	·634159
	12	−·002595	−·008435	−·016736	−·031865	−·070766	−·302716
$\sigma = 2$	2	·570345	−·190115	—	—	—	—
	4	·082322	·452772	−·288127	—	—	—
	6	·030056	·110206	·390729	−·330617	—	—
	8	·014883	·049610	·109141	·354708	−·354708	—
	10	·008642	·027689	·053878	·010432	·331293	−·370269
	12	·005541	·017393	·032067	·054078	·099433	·314872
$\sigma = 4$	4	·544509	−·272254	·054451	—	—	—
	6	·164624	·340222	−·406071	·120724	—	—
	8	·078046	·197717	·176905	−·439661	·169100	—
	10	·044511	·120817	·153246	·079485	−·443206	·205388
	12	·028277	·079333	·112323	·107893	·020297	−·436380
$\sigma = 6$	6	·520106	−·312062	·104021	−·014860	—	—
	8	·202631	·243157	−·428420	·228684	−·043421	—
	10	·110160	·209079	·010566	−·396801	·309572	−·072616
	12	·068548	·154233	·103638	−·104911	−·331186	·360336
$\sigma = 8$	8	·499435	−·332957	·142696	−·035674	·003964	—
	10	·222513	·168020	−·410966	·300467	−·103688	·014380
	12	·131741	·193952	−·098457	−·300774	·376055	−·169556
$\sigma = 10$	10	·481977	−·344269	·172134	−·057379	·011476	−·001043
	12	·233552	·110288	−·378441	·341678	−·162386	·041481
$\sigma = 12$	12	·467044	−·350283	·194601	−·077841	·021229	−·003538

TABLE VIII.—$s^{\sigma}, \frac{1}{2\epsilon_{\sigma}}\int_{-1}^{+1} R^{\sigma}_{n} \sin p\theta \, d\mu$; $\epsilon_0 = 1$; $\epsilon_{\sigma} = 2$ if $\sigma > 0$.

	n; p:	2.	4.	6.	8.	10.	12.
$\sigma = 0$	1	·680175	—	—	—	—	—
	3	− ·259746	·649365	—	—	—	—
	5	− ·508764	− ·284908	·641042	—	—	—
	7	− ·020794	− ·062381	− ·294084	·637181	—	—
	9	− ·010970	− ·027582	− ·067230	− ·298801	·634953	—
	11	− ·006638	− ·015410	− ·030821	− ·069860	− ·301668	·633503
$\sigma = 2$	3	·502996	− ·251498	—	—	—	—
	5	·104266	·417062	− ·312797	—	—	—
	7	·043315	·110695	·370588	− ·344117	—	—
	9	·023011	·052596	·106836	·341875	− ·363242	—
	11	·013975	·030445	·054236	·101817	·322420	− ·376156
$\sigma = 4$	5	·451483	− ·361187	·090297	—	—	—
	7	·203168	·248316	− ·428909	·146732	—	—
	9	·110869	·177390	·121956	− ·443476	·188477	—
	11	·068190	·119333	·129562	·046369	− ·440509	·220254
$\sigma = 6$	7	·402871	− ·402871	·172659	− ·028776	—	—
	9	·239504	·109488	− ·420843	·273719	− ·058165	—
	11	·151841	·157264	− ·058567	− ·365287	·337956	− ·086550
$\sigma = 8$	9	·362831	− ·414664	·233249	− ·069111	·008639	—
	11	·251159	·008970	− ·364778	·346838	− ·137539	·020930
$\sigma = 10$	11	·330211	− ·412764	·275176	− ·110070	·025016	− ·002502

TABLE IX.—$\text{Log}\, v_n^\sigma = \log \{\frac{1}{4}\int_{-1}^{+1} \text{R}_n^\sigma \cos p\theta \, d\mu$.

	n ; p :	1.	3.	5.	7.	9.	11.
$\sigma=1$	2	9·5800455	−9·5800455	—	—	—	—
	4	9·0660312	9·4639712	−9·6100992	—	—	—
	6	8·8018414	8·9479694	9·4250907	−9·6213853	—	—
	8	8·6181138	8·6972950	8·8935897	9·4054731	−9·6273218	—
	10	8·4761883	8·5263549	8·6402982	8·8621470	9·3936260	−9·6309869
	12	8·3602405	8·3950026	8·4707233	8·6042622	8·8416231	9·3856911
$\sigma=3$	4	9·4885802	−9·6646714	9·1875502	—	—	—
	6	9·2560839	8·7789626	−9·6635691	9·3432340	—	—
	8	9·0834581	8·9063668	−8·4292455	−9·7192802	9·4182502	—
	10	8·9451143	8·8459828	8·4452498	−8·8051465	−9·6157008	9·4630907
	12	8·8316471	8·7667063	8·5739550	−7·1536109	−8·9094858	−9·5937325
$\sigma=5$	6	9·4105833	−9·6658558	9·4105833	−8·7116133	—	—
	8	9·2732328	−8·8752928	−9·5742628	9·5519864	−8·9722028	—
	10	9·1484712	8·3033732	−9·2224513	−9·4272248	9·6057041	−9·1136057
	12	9·0410838	8·6608727	−8·8070007	−9·2433905	−9·2576896	9·6280460
$\sigma=7$	8	9·3462968	−9·6473268	9·5011988	−9·0452668	8·2001688	—
	10	9·2593087	−9·1818966	−9·4294507	9·6045424	−9·2974665	8·5468363
	12	9·1654792	−8·5634192	−9·3239854	−8·9471115	9·6059385	−9·4247544
$\sigma=9$	10	9·2921481	−9·6231415	9·5439602	−9·2129670	8·6231415	−7·6688990
	12	9·2367633	−9·2932447	−9·2381017	9·5966222	−9·4438888	8·9642644
$\sigma=11$	12	9·2455204	−9·5977029	9·5642792	−9·3124671	8·8573402	−8·1663392

The minus sign has been inserted where v_n^σ is negative.

TABLE X.—$\text{Log}\, u_n^\sigma = \log \tfrac{1}{4}\int_{-1}^{+1} \text{R}_n^\sigma \cos p\theta \, d\mu$.

	n; p:	0.	2.	4.	6.	8.	10.	12.
$\sigma = 1$	1	9·8326206	−9·5315906	—	—	—	—	—
	3	9·2015646	9·5025946	−9·5995046	—	—	—	—
	5	8·8966219	8·9935319	9·4406899	−9·6167812	—	—	—
	7	8·6912647	8·7404827	8·9165740	9·4138986	−9·6247520	—	—
	9	8·5358191	8·5657824	8·6651669	8·8760203	9·3988990	−9·6293480	—
	11	8·4105944	8·4307978	8·4955231	8·6204618	8·8509108	9·3893017	−9·6323398
$\sigma = 3$	3	9·7896102	−9·6135189	9·0114589	—	—	—	—
	5	9·4072166	9·1731333	−9·6704580	9·2822778	—	—	—
	7	9·1850978	9·0881877	7·8329152	−9·6524592	9·3861912	—	—
	9	9·0230421	8·9679945	8·7008228	−8·6898274	−9·6276795	9·4431551	—
	11	8·8945021	8·8585895	8·7158180	8·0494041	−8·8696784	−9·6043640	9·4794253
$\sigma = 5$	5	9·7567015	−9·6317628	9·2338228	−8·4556715	—	—	—
	7	9·4505217	8·6546417	−9·6323653	9·4997397	−8·8654951	—	—
	9	9·2683891	8·9673591	−9·1392944	−9·5042781	9·5846591	−9·0516800	—
	11	9·1313325	8·9552412	−8·4858068	−9·2468638	−9·3449135	9·6193457	−9·1634137
$\sigma = 7$	7	9·7310723	−9·6341623	9·3331323	−8·7890643	7·8859743	—	—
	9	9·4629721	8·3168441	−9·5651678	9·5738366	−9·1960220	8·4011650	—
	11	9·3046393	8·7605713	−9·3112364	−9·2411257	9·6122664	−9·3703182	8·6591137
$\sigma = 9$	9	9·7102198	−9·6310386	9·3880006	−8·9620318	8·3088193	−7·3088193	—
	11	9·4651149	8·8119024	−9·4869603	9·5951933	−9·3544907	8·8221216	−7·9088124
$\sigma = 11$	11	9·6926784	−9·6257316	9·4216116	−9·0694291	8·5465504	−7·8061877	6·7270064

The minus sign has been inserted where u_n^σ is negative.

TABLE XI.—$\log m_n^\sigma = \log \frac{1}{2\epsilon_\sigma}\int_{-1}^{+1} \mathrm{R}_n^\sigma \cos p\theta \, d\mu \;.\; \epsilon_\sigma = 1\;;\; \epsilon_\sigma = 2 \text{ if } \sigma > 0.$

	$n\,;\,p$:	1.	3.	5.	7.	9.	11.
$\sigma = 0$	0	9·8950899	—	—	—	—	—
	2	- 9·3414849	9·8186062	—	—	—	—
	4	- 8·5660312	- 9·4410925	9·8090692	—	—	—
	6	- 8·1407316	- 8·7639809	- 9·4629509	9·8053736	—	—
	8	- 7·8399625	- 8·3962650	- 8·8144084	- 9·4724198	9·8034130	—
	10	- 7·6060069	- 8·1332947	- 8·4690868	- 8·8370636	- 9·4776870	9·8021982
	12	- 7·4141931	- 7·9260765	- 8·2236460	- 8·5033129	- 8·8498183	- 9·4810365
$\sigma = 2$	2	9·7561369	- 9·2790156	—	—	—	—
	4	8·9155162	9·6558778	- 9·4595842	—	—	—
	6	8·4779326	9·0422040	9·5918759	- 9·5193252	—	—
	8	8·1726861	8·6955648	9·0379875	9·5498708	- 9·5498708	—
	10	7·9365977	8·4422012	8·7314128	8·0183564	9·5202127	- 9·5685174
	12	7·7436014	8·2403773	8·5060576	8·7330181	8·9975313	9·4981337
$\sigma = 4$	4	9·7360052	- 9·4349752	8·7360052	—	—	—
	6	9·2164933	9·5317637	- 9·6086037	9·0817947	—	—
	8	8·8923524	9·2960447	9·2477400	- 9·6431178	9·2281445	—
	10	8·6484713	9·0821269	9·1853903	8·9002833	- 9·6466060	9·3125758
	12	8·4514358	8·8994547	9·0504693	9·0329935	8·3074257	- 9·6398642
$\sigma = 6$	6	9·7160907	- 9·4942420	9·0171207	- 8·1720227	—	—
	8	9·3067061	9·3858874	- 9·6318698	9·3592352	- 8·6376994	—
	10	9·0420242	9·3203110	8·0239261	- 9·5985729	9·4907621	- 8·8610307
	12	8·8359967	9·0881793	9·0155212	- 9·0208233	- 9·5200723	9·5567071
$\sigma = 8$	8	9·6984793	- 9·5223881	9·1544113	- 8·5523513	7·5981088	—
	10	9·3473544	9·2253601	- 9·6138070	9·4777977	- 9·0157277	8·1577608
	12	9·1197216	9·2876950	- 8·9932483	- 9·4782406	9·4752512	- 9·2293127
$\sigma = 10$	10	9·6830259	- 9·5368979	9·2358679	- 8·7587466	8·0597766	- 7·0183840
	12	9·3683840	9·0425304	- 9·5779983	9·5336173	- 9·2105476	8·6178501
$\sigma = 12$	12	9·6693581	- 9·5444194	9·2891469	- 8·8912068	8·3269355	- 7·5487843

The minus sign has been inserted where m_n is negative.

TABLE XII.—$\log s_n^\sigma = \log \frac{1}{2\epsilon_\sigma} \int_{-1}^{+1} \mathrm{R}_n^\sigma \sin p\theta \, d\mu$; $\epsilon_0 = 1$; $\epsilon_\sigma = 2$ if $\sigma > 0$.

	n ; p :	2.	4.	6.	8.	10.	12.
$\sigma=0$	1	9·8326206	—	—	—	—	—
	3	−9·4145489	9·8124889	—	—	—	—
	5	−8·7065163	−9·4547043	9·8068868	—	—	—
	7	−8·3179337	−8·7950550	−9·4684709	9·8042629	—	—
	9	−8·0402060	−8·4406207	−8·8275653	−9·4753827	9·8027417	—
	11	−7·8220558	−8·1978112	−8·4888412	−8·8442288	−9·4795297	9·8017490
$\sigma=2$	3	9·7015647	−9·4005347	—	—	—	—
	5	9·0181410	9·6202009	−9·4952622	—	—	—
	7	8·6366429	9·0441281	9·5688910	−9·5367063	—	—
	9	8·3619324	8·7209543	9·0287177	9·5338677	−9·5601966	—
	11	8·1453468	8·4835186	8·7342851	9·0078189	9·5084213	−9·5753680
$\sigma=4$	5	9·6546416	−9·5577315	8·9556716	—	—	—
	7	9·3078542	9·3950043	−9·6323652	9·1665250	—	—
	9	9·0448101	9·2489300	9·0862027	−9·6468700	9·2752590	—
	11	8·8337222	9·0767602	9·1124758	8·6662311	−9·6439547	9·3429247
$\sigma=6$	7	9·6051663	−9·6051663	9·1371896	−8·4590383	—	—
	9	9·3793134	9·0393654	−9·6241205	9·4373054	−8·7646641	—
	11	9·1813900	9·1966300	−8·7676558	−9·5626341	9·5288595	−8·9372649
$\sigma=8$	9	9·5596417	−9·6176967	9·3678192	−8·8395454	7·9364557	—
	11	9·3998851	7·9527901	−9·5620287	9·5401268	−9·1384267	8·3207669
$\sigma=10$	11	9·5187917	−9·6157017	9·4396104	−9·0416704	8·3982178	−7·3982178

The minus sign has been inserted where s_n^σ is negative.